Math 579 Exam 7 Solutions

1. How many three-letter words do not have consecutive identical letters?

   INDIRECT ANSWER: Ω is the set of all words; $|\Omega| = 26^3 = 17576$. Set A to be those words where the first two letters are the same; there are $26^2 = 676$ of these. Set B to be those words where the last two letters are the same; there are 676 of these as well. $A \cap B$ are those words where all three letters are the same; there are 26 of these. Hence, the answer is $17576 - 676 - 676 + 26 = 16250$.

   DIRECT ANSWER: The first letter can be any of 26. The second letter may be any letter but the first. The third letter may be any letter but the second. Hence, the answer is $26 \times 25 \times 25 = 16250$.

2. How many positive integers less than or equal to $1001 = 7 \times 11 \times 13$ are relatively prime to 1001?

   $\Omega = [1001]$. The three properties are 'divisible by 7', 'divisible by 11', and 'divisible by 13'. Fortunately, all the ratios are integers, so the floor function may be omitted. The answer is $1001 - \frac{1001}{7} - \frac{1001}{11} - \frac{1001}{13} + \frac{1001}{7\times11} + \frac{1001}{7\times13} + \frac{1001}{11\times13} - \frac{1001}{7\times11\times13} = 1001 - (11 \times 13) - (7 \times 13) - (7 \times 11) + 13 + 11 + 7 - 1 = 720$.

3. How many three-digit positive integers are divisible by at least one of six and eight?

   This problem is solved in two parts: $\Omega^+ = [999]$ and $\Omega^- = [99]$. The answer is obtained by subtracting the second from the first. Note that to be divisible by both 6 and 8 a number must be divisible by LCM(6,8)=24.

   $\Omega^+ : \lfloor\frac{999}{6}\rfloor + \lfloor\frac{999}{8}\rfloor - \lfloor\frac{999}{24}\rfloor = 166 + 124 - 41 = 249$.
   $\Omega^- : \lfloor\frac{99}{6}\rfloor + \lfloor\frac{99}{8}\rfloor - \lfloor\frac{99}{24}\rfloor = 16 + 12 - 4 = 24$.

   Putting it all together, the answer is $249 - 24 = 225$.

4. Show an example of three subsets A, B, C of the natural numbers $\mathbb{N}$ so that:

   (a) $|A \cap B| = |A \cap C| = |B \cap C| = \infty$
   (b) $|A \cap B \cap C| = 0$
   (c) $A \cup B \cup C = \mathbb{N}$

   Let $X_0 = \{3n|n \in \mathbb{N}\}, X_1 = \{3n+1|n \in \mathbb{N}\}, X_2 = \{3n+2|n \in \mathbb{N}\}$. Note that $X_0 \cup X_1 \cup X_2 = \mathbb{N}$, but that they are pairwise disjoint.

   Set $A = \{1, 2, 4, 5, 7, 8 \ldots\} = X_1 \cup X_2$. Set $B = \{1, 3, 4, 6, 7, 9 \ldots\} = X_0 \cup X_1$. Set $C = \{2, 3, 5, 6, 8, 9, \ldots\} = X_0 \cup X_1$. Any two of A, B, C have one of the X's in common, so their pairwise intersection is infinite. However, there is no integer in all three: $A \cap B = X_1$, but X_1 is disjoint from C.

   Any other partition of $\mathbb{N}$ into X_0, X_1, X_2 will work similarly.

5. (12 points) Four married couples (8 people) get in one line at a buffet. To be sociable, they decide that no two married people will stand next to each other in line. How many ways can this be done?

   Let Ω be all orderings of eight people. Set $P = \{P_1, P_2, P_3, P_4\}$, where P_i is the property where couple i are standing together. We seek $f_=(\emptyset)$. To find this, we will calculate $f_\geq(T)$ for all possible $T \subseteq P$. $f_\geq(\emptyset) = |\Omega| = 8!$.

   If $|T| = 1$, then one couple will stand as a block; there are then seven units (six individuals and one block). There are also two orderings for the couple within the block. Hence $f_\geq(T) = 2 \times 7!$ in this case. There are $\binom{4}{1}$ ways to have $|T| = 1$ (four ways to choose one of the four couples).

   If $|T| = 2$, then two couples will each stand as blocks. There are six units, and 2^2 orderings within the two blocks. Hence $f_\geq(T) = 2^2 \times 6!$. There are $\binom{4}{2}$ ways to pick two couples.

   If $|T| = 3$, then three couples will be blocks. There are five units, and 2^3 orderings within the three blocks. Hence, $f_\geq(T) = 2^3 \times 5!$. There are $\binom{4}{3}$ ways to pick three couples.

   If $|T| = 4$, then each couple will be a block. There are 4! ways to order the four blocks, and 2^4 ways to order within each of the blocks. Hence, $f_\geq(T) = 2^4 \times 4!$. There is $\binom{4}{4}$ way to pick the four couples.

   Putting it all together, the answer is $8! - \binom{4}{1}2^1 7! + \binom{4}{2}2^2 6! - \binom{4}{3}2^3 5! + \binom{4}{4}2^4 4! = 40320 - 40320 + 17280 - 3840 + 384 = 13824$. Some of you interpreted the question to have the members of each couple be indistinguishable; that is, you counted expressions such as ABCDCDAB where there are no two consecutive identical letters. This is a strange interpretation; however if you fully justified this choice I awarded full credit.

Exam statistics: Low grade=30(D-); Median grade=39(C+); High grade=53(A+)