MATH 579: Combinatorics
Homework 6: Due Oct.16

Please solve these problems using the methods of difference calculus (as presented in class).

1. Prove the following properties for arbitrary constant C and functions $f(x), g(x)$.
   (a) $\Delta C = 0$;
   (b) $\Delta(Cf(x)) = C\Delta f(x)$; and
   (c) $\Delta(f(x) + g(x)) = (\Delta f(x)) + (\Delta g(x))$.
2. Find all functions $f(x)$ satisfying $\Delta(\Delta f(x)) = 3$.
3. Compute $\sum_{i=1}^{n} i^5$, for arbitrary $n \in \mathbb{N}$.
4. Let $c \in \mathbb{R}$. Compute Δc^x. Use this to find an anti-difference of c^x, and hence the geometric sum $\sum_a^b c^x \delta x$ (for $c \neq 1$).
5. For $c \in \mathbb{R}$ and $x \in \mathbb{N}$, compute $\Delta c^{\underline{x}}$. Use this to find an anti-difference of $\frac{(-2)^{\underline{k}}}{k}$, and hence the sum $\sum_{k=2}^{n} \frac{(-2)^{\underline{k}}}{k}$.
6. For $k \in \mathbb{N}$, we define $x^{\underline{-k}} = \frac{1}{(x+1)(x+2)\cdots(x+k)}$. Prove that $\Delta x^{\underline{-k}} = -kx^{\underline{-k-1}}$.
7. For $x \in \mathbb{N}$, we define $H_x = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}$. Henceforth we may consider H_x to be a basic function, in "closed form". Prove that $\Delta H_x = x^{\underline{-1}}$.
8. Prove that $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$ for all integers m, n. (there are cases)
9. Calculate $\displaystyle\sum_{0}^{n} x3^x \delta x$. Your answer should be a function of n.
10. Calculate $\displaystyle\sum_{0}^{n} x^2 2^x \delta x$.
11. Calculate $\displaystyle\sum_{0}^{n} xH_x \delta x$. (hint: summation by parts and exercise 8)
12. Calculate $\displaystyle\sum_{1}^{n} \frac{2k+1}{k(k+1)}$.