MATH 579: Combinatorics

Homework 5: Due Oct.9

In the following, assume that x, y are arbitrary real (or complex) numbers, while a, b, c represent nonnegative integers. Your task is to prove each of the following:

1. (Symmetry) $\binom{a+b}{a} = \binom{a+b}{b}$.
2. (Pascal's Rule) $\binom{x}{a} + \binom{x}{a+1} = \binom{x+1}{a+1}$.
3. (Extraction) $\binom{x}{a} = \frac{x}{a}\binom{x-1}{a-1}$. (provided $a \neq 0$)
4. (Committee/Chair) $(a+1)\binom{x}{a+1} = x\binom{x-1}{a}$.
5. (Twisting) $\binom{x}{a}\binom{x-a}{b} = \binom{x}{b}\binom{x-b}{a}$.
6. (Negation) $\binom{x}{a} = (-1)^a\binom{a-x-1}{a}$.
7. $\binom{-\frac{1}{2}}{a} = (-1)^a\binom{2a}{a}2^{-2a}$.
8. $\binom{\frac{1}{2}}{a} = (-1)^{a+1}\binom{2a}{a}\frac{2^{-2a}}{2a-1}$.
9. (Chu-Vandermonde) $$\binom{x+y}{a} = \sum_{k=0}^{a}\binom{x}{k}\binom{y}{a-k}.$$ Hint: $(t+1)^x(t+1)^y$
10. (Chu-Vandermonde II) $$(x+y)^{\underline{a}} = \sum_{k=0}^{a}\binom{a}{k}x^{\underline{k}}y^{\underline{a-k}}.$$
11. $\sum_{k=0}^{a}\binom{a}{k}^2 = \binom{2a}{a}$. Hint: Chu-Vandermonde
12. (Hockey Stick) $$\sum_{k=a}^{a+b}\binom{k}{a} = \binom{a+b+1}{a+1}.$$
13. Suppose that $b \leq \frac{a-1}{2}$. Then $\binom{a}{b} \leq \binom{a}{b+1}$.
14. Suppose that $b \geq \frac{a-1}{2}$. Then $\binom{a}{b} \geq \binom{a}{b+1}$. Hint: Symmetry identity
15. $\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n$. Hint: $(1+1)^{2n}$