MATH 579: Combinatorics

Homework 5 Solutions

1. (Symmetry) $\binom{a+b}{a} = \binom{a+b}{b}$.
Since $a+b, a \in \mathbb{Z}$ with $a+b \geq a \geq 0$, we may use the factorial form of the binomial coefficient. $\binom{a+b}{a} = \frac{(a+b)!}{a!((a+b)-a)!} = \frac{(a+b)!}{a!b!} = \frac{(a+b)!}{b!a!} == \frac{(a+b)!}{b!((a+b)-b)!} = \binom{a+b}{b}$.

2. (Pascal's Rule) $\binom{x}{a} + \binom{x}{a+1} = \binom{x+1}{a+1}$.
We calculate $(a+1)x^{\underline{a}} + x^{\underline{a+1}} = x^{\underline{a}}\big((a+1) + (x-(a+1)+1)\big) = x^{\underline{a}}(x+1) = (x+1)^{\underline{a+1}}$. Divide both sides by $(a+1)!$ and the result follows.

3. (Extraction) $\binom{x}{a} = \frac{x}{a}\binom{x-1}{a-1}$. (provided $a \neq 0$)
Peeling off the first term, we see that $x^{\underline{a}} = x \cdot (x-1)^{\underline{a-1}}$. Divide both sides by $a! = a \cdot (a-1)!$ and the result follows.

4. (Committee/Chair) $(a+1)\binom{x}{a+1} = x\binom{x-1}{a}$.
This symmetric version of the extraction identity comes from multiplying both sides by a, and replacing a by $a+1$. It gets its name from the special case when $x \in \mathbb{N}$. Then, the LHS counts the ways to pick a committee of $a+1$ out of x people, then pick a chair from the committee's members. The RHS counts the ways to pick the chair first, out of x people, then pick the remaining a members of the committee out of the remaining $x-1$ people.

5. (Twisting) $\binom{x}{a}\binom{x-a}{b} = \binom{x}{b}\binom{x-b}{a}$.
We see that $x^{\underline{a}}(x-a)^{\underline{b}} = x(x-1)\cdots(x-a+1)(x-a)(x-a-1)\cdots(x-a-b+1) = x^{\underline{a+b}} = x(x-1)\cdots(x-b+1)(x-b)(x-b-1)\cdots(x-b-a+1) = x^{\underline{b}}(x-b)^{\underline{a}}$. Divide both sides by $a!b! = b!a!$ and the result follows.

6. (Negation) $\binom{x}{a} = (-1)^a\binom{a-x-1}{a}$.
We write $x^{\underline{a}} = (x-0)(x-1)(x-2)\cdots(x-a+2)(x-a+1) = (-1)^a(0-x)(1-x)(2-x)\cdots(a-x-2)(a-x-1) = (-1)^a(a-x-1)(a-x-2)\cdots(2-x)(1-x)(0-x) = (-1)^a(a-x-1)(a-x-2)\cdots(a-x-1-(a-3))(a-x-1-(a-2))(a-x-1-(a-1)) = (-1)^a(a-x-1)^{\underline{a}}$. Divide both sides by $a!$ and the result follows.

7. $\binom{-\frac{1}{2}}{a} = (-1)^a\binom{2a}{a}2^{-2a}$.
For this problem and the next it is useful (but not necessary) to define the double factorial, $n!! = n \cdot (n-2)!!$, with $0!! = 1!! = 1$. We now prove a lemma: For $n = 2k-1$ odd, $n!! = \frac{(2k)!}{2^k k!}$. Proof: Induction on k. $k = 1$, $1!! = 1 = \frac{2!}{2^1 1!}$. Assume that $n!! = \frac{(2k)!}{2^k k!}$, and multiply both sides by $n+2 = 2k+1$. We get $(n+2)!! = (n+2) \cdot n!! = \frac{(2k+1)\cdot(2k)!}{2^k k!} = \frac{(2k+2)(2k+1)\cdot(2k)!}{(2k+2)2^k k!} = \frac{(2(k+1))!}{2^{k+1}(k+1)!}$.
Now, $(-\frac{1}{2})^{\underline{a}} = (-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-a+1) = (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2a-1}{2}) = (-1)^a 2^{-a}(2a-1)!! = (-1)^a 2^{-a}\frac{(2a)!}{2^a a!} = (-1)^a 2^{-2a}\frac{(2a)!}{a!}$. Now divide both sides by $a!$.

8. $\binom{\frac{1}{2}}{a} = (-1)^{a+1}\binom{2a}{a}\frac{2^{-2a}}{2a-1}$.
We have $(\frac{1}{2})^{\underline{a}} = (\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-a+1) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2a-3}{2}) = (-1)^{a-1}2^{-a}(2a-3)!! = (-1)^{a-1}2^{-a}\frac{(2a-2)!}{2^{a-1}(a-1)!} = (-1)^{a-1}2^{-a}(2a-3)!! = (-1)^{a-1}2^{-a}\frac{(2a)(2a-1)(2a-2)!}{(2a)(2a-1)2^{a-1}(a-1)!} =$
$= (-1)^{a-1}2^{-a}\frac{(2a)!}{2^a a!(2a-1)} = (-1)^{a+1}2^{-2a}\frac{1}{2a-1}\frac{(2a)!}{a!}$. Now divide both sides by $a!$.

9. (Chu-Vandermonde) $\displaystyle\binom{x+y}{a} = \sum_{k=0}^{a} \binom{x}{k}\binom{y}{a-k}$. Hint: $(t+1)^x(t+1)^y$

Assuming $|t| < 1$, we apply Newton's binomial theorem three times as follows. $\sum_{a\geq 0}\binom{x+y}{a}t^a = (t+1)^{x+y} = (t+1)^x(t+1)^y = \left(\sum_{a\geq 0}\binom{x}{a}t^a\right)\left(\sum_{a\geq 0}\binom{y}{a}t^a\right) = \sum_{a\geq 0}\left(\sum_{k=0}^{a}\binom{x}{k}\binom{y}{a-k}\right)t^a$, using the formula for the product of power series. We now equate coefficients of t^a and are done.

10. (Chu-Vandermonde II) $\displaystyle(x+y)^{\underline{a}} = \sum_{k=0}^{a}\binom{a}{k}x^{\underline{k}}y^{\underline{a-k}}$.

Multiply both sides of the Chu-Vandermonde identity by $a!$ and note that $a!\binom{x}{k}\binom{y}{a-k} = \frac{a!}{k!(a-k)!}x^{\underline{k}}y^{\underline{a-k}} = \binom{a}{k}x^{\underline{k}}y^{\underline{a-k}}$.

11. $\sum_{k=0}^{a}\binom{a}{k}^2 = \binom{2a}{a}$. Hint: Chu-Vandermonde

Apply Chu-Vandermonde with $x = y = a$. Note that, by the symmetry identity, $\binom{a}{a-k} = \binom{a}{k}$.

12. (Hockey Stick) $\displaystyle\sum_{k=a}^{a+b}\binom{k}{a} = \binom{a+b+1}{a+1}$.

Induction on b. If $b = 0$, the LHS is $\binom{a}{a} = 1 = \binom{a+1}{a+1}$. Suppose now that $\sum_{k=a}^{a+b}\binom{k}{a} = \binom{a+b+1}{a+1}$, and add $\binom{a+b+1}{a}$ to both sides. We have $\sum_{k=a}^{a+b+1}\binom{k}{a} = \binom{a+b+1}{a} + \binom{a+b+1}{a+1} = \binom{a+b+2}{a+1}$, applying Pascal's Rule.

13. Suppose that $b \leq \frac{a-1}{2}$. Then $\binom{a}{b} \leq \binom{a}{b+1}$.

We have $a \geq 2b+1$, hence $a-b \geq b+1$, hence $\frac{1}{b+1} \geq \frac{1}{a-b}$. We multiply both sides by $\frac{a!}{b!(a-b-1)!}$ to get $\frac{a!}{(b+1)!(a-b-1)!} \geq \frac{a!}{b!(a-b)!}$, the desired result.

14. Suppose that $b \geq \frac{a-1}{2}$. Then $\binom{a}{b} \geq \binom{a}{b+1}$.

Set $b' = a-(b+1)$. Since $b \geq \frac{a-1}{2}$, $b+1 \geq \frac{a+1}{2}$, and hence $b' = a-(b+1) \leq a - \frac{a+1}{2} = \frac{a-1}{2}$. Apply the previous problem to get $\binom{a}{b'} \leq \binom{a}{b'+1}$. Apply the symmetry identity twice to get $\binom{a}{a-b'} \leq \binom{a}{a-b'-1}$, which is the desired result since $a-b' = b+1$ and $a-b'-1 = b$.

This problem, and the previous, prove that each row of Pascal's triangle is nondecreasing until the middle, and then nonincreasing. Such sequences are called unimodal.

15. $\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n$. Hint: $(1+1)^{2n}$

We have $4^n = (1+1)^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}$, by Newton's binomial theorem. Since all the summands are nonnegative, if we replace all but $\binom{2n}{n}$ with zero, the sum only decreases: $4^n = \sum_{i=0}^{2n}\binom{2n}{i} \geq \binom{2n}{n}$. This gives the upper bound. By unimodality proved by the previous two problems, the largest summand is $\binom{2n}{n}$. Hence if we replace each summand by this largest one, the sum only increases: $4^n = \sum_{i=0}^{2n}\binom{2n}{i} \leq (2n+1)\binom{2n}{n}$. Dividing by $2n+1$ gives the lower bound.