MATH 579: Combinatorics
Homework 4 Solutions

1. Let $n \in \mathbb{N}_0$. Prove that $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

We apply the binomial theorem with $x = y = 1$ to get $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{(n-i)} = \sum_{i=0}^{n} \binom{n}{i}$.

2. Let $n \in \mathbb{N}_0$. Prove that $\frac{3^n + (-1)^n}{2} = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} 2^i \binom{n}{i}$.

Apply the binomial theorem with $x = 2, y = 1$ to get $3^n = \sum_{i=0}^{n} \binom{n}{i} 2^i$. Apply the binomial theorem with $x = -2, y = 1$ to get $(-1)^n = \sum_{i=0}^{n} \binom{n}{i}(-2)^i$. Adding, we get $3^n + (-1)^n = \sum_{i=0}^{n} (2^i + (-2)^i)\binom{n}{i}$. Note that $2^i + (-2)^i = \begin{cases} 0 & i \text{ odd} \\ 2 \cdot 2^i & i \text{ even} \end{cases}$. Hence $3^n + (-1)^n = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} 2 \cdot 2^i \binom{n}{i}$. Now divide both sides by 2.

3. Let $n \in \mathbb{N}_0$. Prove that $\frac{6^n - (-4)^n}{2} = \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} 5^i \binom{n}{i}$.

Apply the binomial theorem with $x = 5, y = 1$ to get $6^n = \sum_{i=0}^{n} \binom{n}{i} 5^i$. Apply the binomial theorem with $x = -5, y = 1$ to get $(-4)^n = \sum_{i=0}^{n} \binom{n}{i}(-5)^i$. Subtract the second from the first to get $6^n - (-4)^n = \sum_{i=0}^{n} (5^i - (-5)^i)\binom{n}{i}$. Note that $5^i - (-5)^i = \begin{cases} 2 \cdot 5^i & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$. Hence $6^n + (-4)^n = \sum_{\substack{i=0 \\ i \text{ odd}}}^{n} 2 \cdot 5^i \binom{n}{i}$. Now divide both sides by 2, and observe that 0 is not odd.

4. Let $n \in \mathbb{N}_0$. Prove that $n 2^{n-1} = \sum_{i=0}^{n} i \binom{n}{i}$.

Apply the binomial theorem with $y = 1$ to get $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. Take the derivative with respect to x of both sides, to get $n(x+1)^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i x^{i-1}$. Now substitute $x = 1$ to get the desired formula.

5. Let $n \in \mathbb{N}_0$. Prove that $\frac{1}{n+1} = \sum_{i=0}^{n} \frac{(-1)^i}{i+1} \binom{n}{i}$.

Apply the binomial theorem with $y = 1$ to get $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. Take the integral with respect to x of both sides to get $\frac{1}{n+1}(x+1)^{n+1} + C = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i+1} x^{i+1}$. To find C, we plug in $x = 0$. The RHS is 0, while the LHS is $\frac{1}{n+1} + C$. Hence $C = \frac{-1}{n+1}$. Now, we plug in $x = -1$ instead. We get $\frac{-1}{n+1} = 0 + C = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i+1}(-1)^{i+1}$. Lastly, we multiply both sides by -1 to get the desired formula.

6. How many different acronyms does MISSISSIPPI have?
This eleven-letter word contains one M, two P's, four I's, and four S's. This is calculated via the multinomial coefficient $\binom{11}{1,2,4,4} = \frac{11!}{1!2!4!4!} = 34650$.

7. Let $n \in \mathbb{N}_0$. Prove that $3^n = \sum_{i+j+k=n} \binom{n}{i,j,k}$.
We can apply the multinomial theorem with $x = y = z = 1$ to get $3^n = (1+1+1)^n = \sum_{i+j+k=n} \binom{n}{i,j,k} 1^i 1^j 1^k = \sum_{i+j+k=n} \binom{n}{i,j,k}$.

8. Let $n \in \mathbb{N}_0$. Prove that $1 = \sum_{i+j+k=n} (-1)^i \binom{n}{i,j,k}$.
We can apply the multinomial theorem with $y = z = 1$, $x = -1$ to get $1 = 1^n = (-1+1+1)^n = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^i 1^j 1^k = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^i$.

9. What is the largest coefficient in $(x_1 + x_2 + x_3 + x_4 + x_5)^{150}$?
The key to this problem is the following:

Lemma: Let $a, b \in \mathbb{N}_0$. If $a \geq b+2$, then $a!b! > (a-1)!(b+1)!$.
Proof: Since $a \geq b+2$, in fact $a > b+1$. Now multiply both sides by $(a-1)!b!$.

Now, the coefficients in our multivariate polynomial are all $\binom{150}{a_1,a_2,a_3,a_4,a_5} = \frac{150!}{a_1!a_2!a_3!a_4!a_5!}$, such that $a_1 + a_2 + a_3 + a_4 + a_5 = 150$. If $a_1 \geq a_2 + 2$, then we can replace the variables $\{a_1, a_2\}$ by $\{a_1', a_2'\}$ where $a_1' = a_1 - 1$ and $a_2' = a_2 + 1$. We have $a_1' + a_2' + a_3 + a_4 + a_5 = 150$, so this gives another coefficient. By the lemma, our denominator has strictly decreased, so this coefficient is strictly larger. By applying this reasoning symmetrically to every pair of variables (not just a_1, a_2), we know that no variable can be 2 or more larger than any other. Hence our largest coefficient must arise where all the variables (a_i's) are equal, or within 1. As it happens, we can take $a_1 = a_2 = a_3 = a_4 = a_5 = 30$; this must be the maximal coefficient.