MATH 579: Combinatorics
Homework 3: Due Sep. 18

1. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1 + x_2 + x_3 + \cdots + x_n = 26$, with each $x_i \geq 0$. Your answer should be a function of n.

2. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1 + x_2 + x_3 + \cdots + x_n = 26$, with each $x_i \geq -2$. Your answer should be a function of n.

3. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1 + x_2 + x_3 + \cdots + x_n = 26$, with each $x_i \geq 1 + i$. Your answer should be a function of n.

4. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1 + x_2 + x_3 + \cdots + x_n = 26$, with each $x_i \geq i^2$. Your answer should be a function of n.

5. Find explicitly all integer partitions of 8. Match each partition with its conjugate, and give its rank.

6. Find explicitly all integer partitions of 10 into odd parts.

7. Find explicitly all integer partitions of 10 into distinct parts.

8. Find explicitly all integer partitions of 25 into distinct odd parts.

9. Find explicitly all self-conjugate integer partitions of 25, and match them with your answers from exercise 8.

10. Prove that $p(1) + p(2) + \cdots + p(n) < p(2n)$ for all $n \in \mathbb{N}$.
    Hint: Look at the largest part.

11. Let $v(n)$ denote the number of integer partitions of n in which each part is at least 2. Prove that $v(n) = p(n) - p(n-1)$, for all $n \geq 2$.
    Hint: Look at the smallest part.

12. Let $n, k \in \mathbb{N}$. Let $p_k(n)$ denote the number of integer partitions of n into exactly k parts. Prove that $p_k(n)$ also counts the number of partitions of n, in which the largest part has size k.

13. Prove that $p_k(n)$ satisfies the recurrence relation $p_k(n) = p_k(n-k) + p_{k-1}(n-1)$.
    Hint: Look at the smallest part.

14. Using the recurrence relation from exercise 13, and the boundary conditions $p_1(n) = 1$ ($\forall n \in \mathbb{N}$) and $p_k(n) = 0$ ($\forall k \in \mathbb{N}, \forall n \in \mathbb{Z}$ with $n < k$), calculate $p_5(10)$.