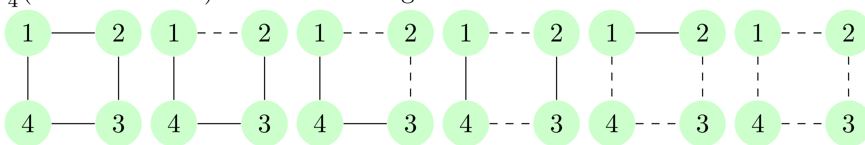
MATH 579: Combinatorics

Homework 11 Solutions

1. Count the number of edge colorings of a square, up to rotation. Give them all explicitly for $m = 2$.

The automorphism group has four elements: id, (1234), (13)(24), (1432). Hence our answer is $\frac{1}{4}(m^4 + m^2 + 2m)$. For $m = 2$ we get six:



2. Count the number of edge colorings of a square, up to rotation and reflection. Give them all explicitly for $m = 2$.

Now the group has eight elements: id, (1234), (13)(24), (1432), (13), (24), (14)(23), (12)(34). Hence our answer is $\frac{1}{8}(m^4 + 2m^3 + 3m^2 + 2m)$. As it happens, for $m = 2$ we still get six, the same as in the previous problem. However, for $m = 3$, we get 21 (but 24 in the previous problem).

3. Count the number of vertex colorings of a tetrahedron, up to rotation and reflection.

The automorphism group was computed in Homework 10, problem 5, to be S_4. This has one element with four cycles (identity), $\binom{4}{2} = 6$ elements with three cycles, $3! = 6$ elements with one cycle (to see this, put in canonical form and order the remaining three elements). Hence there must be $4! - (1 + 6 + 6) = 11$ elements with two cycles. Answer: $\frac{1}{24}(m^4 + 6m^3 + 11m^2 + 6m) = \frac{m(m+1)(m+2)(m+3)}{24}$.

4. Count the number of face colorings of a tetrahedron, up to rotation and reflection.

It turns out that the four faces of a tetrahedron have complete symmetry, just as with the vertices. It has automorphism group S_4, with the same answer as the previous problem.

5. Count the number of vertex colorings of a tetrahedron, up to rotation.

This was found in Homework 10, problem 6, as A_4: $(2, 3, 4), (2, 4, 3), (1, 3, 4), (1, 4, 3), (1, 2, 4), (1, 4, 2), (2, 3, 4), (2, 4, 3), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), id$. Answer: $\frac{1}{12}(m^4 + 11m^2)$

6. Count the number of face colorings of a tetrahedron, up to rotation.

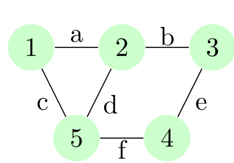
It turns out again that the faces of a tetrahedron have A_4 as automorphism group for rotations, so the answer is the same as the previous problem.

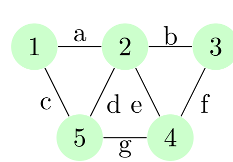
7. Count the number of vertex colorings of the three graphs in problem 7 of the previous homework set (3 parts), up to graph automorphism.

The first one has dihedral group D_5 as its automorphism group: id, (1, 2, 3, 4, 5), (1, 3, 5, 2, 4), (1, 4, 2, 5, 3), (1, 5, 4, 3, 2), (2, 5)(3, 4), (1, 2)(3, 5), (1, 3)(4, 5), (1, 4)(2, 3), (1, 5)(2, 4), with answer $\frac{1}{10}(m^5 + 5m^3 + 4m)$. The second one has group $id, (2, 5)(3, 4)$, with answer $\frac{1}{2}(m^5 + m^3)$. The last has group $id, (1, 3)(4, 5)$ with answer $\frac{1}{2}(m^5 + m^3)$.

8. Count the number of edge colorings of the three graphs in problem 7 of the previous homework set (3 parts), up to graph automorphism.

The first one is again the dihedral group D_5, with the same answer as the previous problem. The next two are more interesting. The second has, again, two automorphisms: $id, (ac)(d)(bf)(e)$, with answer $\frac{1}{2}(m^6 + m^4)$. The third has automorphisms $id, (ab)(cf)(de)(g)$, with answer $\frac{1}{2}(m^7 + m^4)$.





9. Count the number of edge colorings of a tetrahedron, up to rotation.

We get twelve rotations (as in problem 5), but now there are six edges which move. Eight of the rotations are around a line through a vertex, which gives two cycles of length 3. Three of the rotations are around a line through the midpoints of two opposite edges, which gives two fixed edges, and two cycles of length two. Lastly is id. Answer: $\frac{1}{12}(m^6 + 3m^4 + 8m^2)$.

10. Count the number of edge colorings of a tetrahedron, up to rotation and reflection.

We will get twelve automorphisms in all, twelve of which are as in the previous problem. There are six pure reflections, where two vertices are fixed and two others swap. This gives two fixed edges, and two cycles of length two. Lastly there are six rotoreflections, all of which have a cycle of length four and a cycle of length two. Answer: $\frac{1}{24}(m^6 + 9m^4 + 14m^2)$

11. Count the number of vertex colorings of a cube, up to rotation.

This was found in Homework 10, problem 10, as the following group with 24 elements: $(1, 2, 3, 4)(5, 6, 7, 8)$, $(1, 3)(2, 4)(5, 7)(6, 8)$, $(1, 4, 3, 2)(5, 8, 7, 6)$, $(1, 5, 8, 4)(2, 6, 7, 3)$, $(1, 8)(4, 5)(2, 7)(3, 6)$, $(1, 4, 8, 5)(2, 3, 7, 6)$, $(1, 2, 6, 5)(3, 7, 8, 4)$, $(1, 6)(2, 5)(3, 8)(4, 7)$, $(1, 5, 6, 2)(3, 4, 8, 7)$, $(2, 4, 5)(3, 8, 6)$, $(2, 5, 4)(3, 6, 8)$, $(1, 3, 6)(4, 7, 5)$, $(1, 6, 3)(4, 5, 7)$, $(1, 6, 8)(2, 7, 4)$, $(1, 8, 6)(2, 4, 7)$, $(1, 3, 8)(2, 7, 5)$, $(1, 8, 3)(2, 5, 7)$, $(1, 2)(3, 5)(4, 6)(7, 8)$, $(1, 7)(2, 8)(3, 4)(5, 6)$, $(1, 5)(2, 8)(3, 7)(4, 6)$, $(1, 7)(2, 6)(3, 5)(4, 8)$, $(1, 4)(2, 8)(3, 5)(6, 7)$, $(1, 7)(2, 3)(4, 6)(5, 8)$, id. Answer: $\frac{1}{24}(m^8 + 17m^4 + 6m^2)$.

12. Count the number of face colorings of a cube, up to rotation.

Label the faces A, B, C, D, E, F, with A, B, C clockwise around a corner, and $\{A, D\}, \{B, E\}, \{C, F\}$ opposite faces. There are still 24 rotations, as in the previous problem. The axis of rotation could pass through two opposite faces, e.g. A, D, giving $(BCEF), (BE)(CF), (BFEC)$. This can be done in three ways. It can pass through two opposite corners, e.g. ABC, DEF, giving $(ABC)(DEF), (ACB)(DFE)$. This can be done in four ways. It can pass through the centers of two opposite edges, such as AB, DE, giving $(AB)(CF)(DE)$. This can be done in six ways. Lastly, there is id. Answer: $\frac{1}{24}(m^6 + 3m^4 + 12m^3 + 8m^2)$.

13. Count the number of edge colorings of a cube, up to rotation.

There are still 24 rotations, as in the previous two problems. Three have axis of rotation passing through two opposite faces, giving structure: $(abcd)(efgh)(ijkl)$, $(ac)(bd)(eg)(fh)(ik)(jl)$, $(adcb)(ehgf)(ilkj)$. Four have axis of rotation passing through two opposite corners, giving structure $(abc)(def)(ghi)(jkl), (acb)(dfe)(gih)(jlk)$. Six have axis of rotation passing through two opposite edges, giving structure $(ab)(cd)(ef)(gh)(ij)$. Answer: $\frac{1}{24}(m^{12} + 6m^7 + 3m^6 + 8m^4 + 6m^3)$.