Math 579 Fall 2013 Exam 7 Solutions

1. Calculate how many permutations of [5] contain none of the cycles (1 2), (2 3), or (3 4).

   Let $S = \{s_1, s_2, s_3\}$ where s_i denotes that the cycle $(i\ i+1)$ is present. We calculate $f_{\geq}(\emptyset) = 5! = 120$, and $f_{\geq}(s_i) = 3! = 6$, since assuming any one of the cycles there are three remaining elements to permute. We have $f_{\geq}(s_1 s_2) = f_{\geq}(s_2 s_3) = f_{\geq}(s_1 s_2 s_3) = 0$, since if (2 3) is present neither of the other two cycles can be present. However $f_{\geq}(s_1 s_3) = 1$, specifically (1 2)(3 4)(5). We want $f_{=}(\emptyset) = 120 - 6 - 6 - 6 + 1 = 103$.

2. Calculate how many permutations π of [5] satisfy $\pi(1) \neq 2$, $\pi(2) \neq 3$, $\pi(3) \neq 4$, $\pi(4) \neq 5$.

	X			
		X		
			X	
				X

   The diagram at left indicates the forbidden positions. We calculate $r_1 = 4, r_2 = \binom{4}{2} = 6, r_3 = \binom{4}{3} = 4, r_4 = \binom{4}{4} = 1$. The formula we want is $5! - r_1 4! + r_2 3! - r_3 2! + r_4 1! = 53$.

3. Calculate how many permutations of [5] have exactly one fixed point.

   There are five possible fixed points, and the remaining elements form a derangement of four elements, of which there are $D(4) = 9$, so the answer is $5 \times 9 = 45$.

4. Calculate how many ways we can list the digits $\{1, 1, 2, 2, 3, 3, 4\}$ so that two identical digits are not in consecutive positions.

   Let $S = \{s_1, s_2, s_3\}$ where s_i denotes that the two i's are consecutive. We calculate $f_{\geq}(\emptyset) = \frac{7!}{2!2!2!} = 630$, $f_{\geq}(s_i) = \frac{6!}{2!2!} = 180$, $f_{\geq}(s_i, s_j) = \frac{5!}{2!} = 60$, and $f_{\geq}(s_1 s_2 s_3) = 4! = 24$. We want $f_{=}(\emptyset) = 630 - 3(180) + 3(60) - 24 = 246$.

5. Calculate how many ways we can list the digits $\{1, 1, 1, 2, 2, 3, 4\}$ so that two identical digits are not in consecutive positions.

   Let's write $1_A, 1_B, 1_C$ to distinguish the 1's; we will divide by 6 in the end. Let $S = \{s_1, s_2, s_3, r\}$ where s_1 denotes 1_A and 1_B together, s_2 denotes 1_A and 1_C together, s_3 denotes 1_B and 1_C together, r denotes the 2's together. We calculate $f_{\geq}(\emptyset) = \frac{7!}{2!} = 2520$, $f_{\geq}(s_i) = 2\frac{6!}{2!} = 720$ (2 because $1_A 1_B$ or $1_B 1_A$), $f_{\geq}(r) = 6! = 720$, $f_{\geq}(s_i s_j) = 2\frac{5!}{2!} = 120$ (2 because $1_A 1_B 1_C$ or $1_C 1_B 1_A$), $f_{\geq}(s_i r) = 2(5!) = 240$. We can't have all three of s_1, s_2, s_3, but $f_{\geq}(s_i s_j r) = 2(4!) = 48$. Putting it all together, $f_{=}(\emptyset) = 2520 - 3(720) - 720 + 3(120) + 3(240) - 3(48) = 576$. Finally, we use equivalence classes to erase the subscripts, giving a final answer of $\frac{576}{6} = 96$.