Math 579 Fall 2013 Exam 6 Solutions

1. Let $\pi = (2\ 3\ 5)(1\ 4)$, $\sigma = (3\ 4\ 6\ 1)(2\ 5)$. Calculate $\sigma \circ \pi$, $\pi \circ \sigma$, and π^σ.

   $\sigma \circ \pi = (1\ 6)(2\ 4\ 3)$, $\pi \circ \sigma = (4\ 6)(1\ 5\ 3)$, $\pi^\sigma = \sigma \circ \pi \circ \sigma^{-1} = (5\ 4\ 2)(3\ 6)$.

2. Calculate how many permutations on $[n]$ have $1, 3$ in the same cycle, but 2 in a different cycle.

   We instead solve the equivalent (by conjugation) problem of counting when $n, n-1$ are in the same cycle, but $n-2$ is in a different cycle. We write all permutations canonically, and apply the Transition Lemma. The resulting one-line permutation corresponds to one of the right type if the three elements are precisely in the order $n-2, n, n-1$. We can build an equivalence relation among all one-line permutations, by permuting these three elements in place; just one-sixth of them are the ones we want. Hence the answer is $\frac{n!}{6}$, provided $n \geq 3$ (0 otherwise).

3. Calculate how many permutations on $[n]$ have $1, 3$ forming a cycle of length 2, and 2 in a different cycle of length 2.

   We proceed as in problem 2, but this time we need not only the order $n-2, n, n-1$, but we need $n, n-1$ to be in the last two positions in just that order, and $n-2$ to be in the fourth-to-last position. The remaining $n-3$ elements may be placed arbitrarily; hence the answer is $(n-3)!$, provided $n \geq 4$ (0 otherwise).

4. Find a formula involving $p(n)$ for the number of partitions of n in which the three largest parts are equal.

   By taking conjugates, our desired answer is equal to the number of partitions of n in which the smallest part is of size at least 3. We will therefore count partitions in which the smallest part is exactly 1, $a(n)$; then partitions in which the smallest part is exactly 2, $b(n)$, then subtract from $p(n)$. For $a(n)$, remove the part of size 1 and we get a bijection with $p(n-1)$. For $b(n)$, remove the part of size 2 and we get a partition of $n-2$ of size at least 2, which is $p(n-2) - a(n-2) = p(n-2) - p(n-3)$. Putting it all together, our answer is $p(n) - p(n-1) - p(n-2) + p(n-3)$.

5. Calculate how many permutations $p \in S_6$ satisfy $p^6 = 1$.

   Note that cycles of length 1,2,3,6 (and only these) vanish when raised to the sixth power.

   Method 1: We count permutations where all cycles are of the specified lengths. Types and counts:

(6,0,0)	(0,3,0)	(0,0,2)	(3,0,1)	(4,1,0)	(2,2,0)	(0,1,1)	(0,0,0,0,0,1)
$\frac{6!}{6!1^6}$	$\frac{6!}{3!2^3}$	$\frac{6!}{2!3^2}$	$\frac{6!}{3!1!1^3 3^1}$	$\frac{6!}{4!1!1^4 2^1}$	$\frac{6!}{2!2!1^2 2^2}$	$\frac{6!}{1!1!2^1 3^1}$	$\frac{6!}{1!6^1}$
1	15	40	40	15	45	120	120

   The grand total is 396.

   Method 2: We count permutations that contain a forbidden cycle length (4 or 5) then subtract from $6! = 720$. The types and counts are:

(2,0,0,1)	(0,1,0,1)	(1,0,0,0,1)
$\frac{6!}{2!1!1^2 4^1}$	$\frac{6!}{1!1!2^1 4^1}$	$\frac{6!}{1!1!1^1 5^1}$
90	90	144

   The grand total is 324 so our answer is $720 - 324 = 396$.