Math 524 Exam 8 Solutions

All the problems concern the vector space $\mathbb{R}_2[t]$ and the bilinear real symmetric form $\langle f|g\rangle = \int_0^1 f(t)g(t)dt$.

1. Under the standard basis $E = \{1, t, t^2\}$, find the metric G_E.

   G_E is the matrix satisfying $\langle f|g\rangle = [f]_E^T G_E [g]_E$. Setting $e_1 = 1, e_2 = t, e_3 = t^2$, we need to compute $\langle e_i|e_j\rangle$ for every i, j. By symmetry, this will only be 6 integrals, and none of them are difficult. $\langle e_1|e_1\rangle = \int_0^1 1dt = 1$, $\langle e_1|e_2\rangle = \int_0^1 tdt = 1/2$, $\langle e_1|e_3\rangle = \langle e_2|e_2\rangle = \int_0^1 t^2dt = 1/3$, $\langle e_2|e_3\rangle = \int_0^1 t^3dt = 1/4$, $\langle e_3|e_3\rangle = \int_0^1 t^4dt = 1/5$. Putting it all together, we get $G_E = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$. Matrices with this particular structure are called Hilbert matrices.

2. Prove that the above form is a (real) inner product.

   A real inner product requires five properties: linearity in each coordinate, symmetry, positivity, and definiteness. The first three are given for free (or are easy to check by properties of the integral). Definiteness is also easy, since $\langle 0|0\rangle = \int_0^1 0dt = 0$. The only significant issue is positivity.

   Analytic solution: Suppose first that $f^2(a) = b$, for some $a \in (0,1)$, $b > 0$. Then, because polynomials are continuous, there is some interval $[a-\epsilon, a+\epsilon]$, in which $f^2 \geq b/2$. Hence, $\langle f|f\rangle = \int_0^1 f^2(t)dt \geq \int_{a-\epsilon}^{a+\epsilon}(b/2)dt = 2\epsilon b/2 = \epsilon b > 0$. Hence if f is nonzero at ANY point in $(0,1)$, then $\langle f|f\rangle > 0$. On the other hand, if f is zero at every point of $(0,1)$, then it must be the zero polynomial. [proof: by the fundamental theorem of algebra, the only polynomial with infinitely many roots is the zero polynomial].

   Algebraic solution: The form is positive if the matrix G is positive definite. G is symmetric, so by Sylvester's criterion we need only check three determinants. $|1| = 1$. $\begin{vmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{vmatrix} = 1/12$. $|G| = 1/2160$. All are positive, hence G is positive definite.

The last two problems refer to the vectors $u(t) = t - 1, v(t) = t^2 - 1$. Set $V = \text{Span}(u, v) = \{at^2 + bt - (a+b)\}$.

3. Find an orthogonal basis for V.

   $\{u, v\}$ is a basis already, but not an orthogonal one; hence Gram-Schmidt is in order. An orthogonal basis will be $\{u, w\}$, for $w = v - Pr_u|v\rangle = v - \frac{|u\rangle\langle u|}{\langle u|u\rangle}|v\rangle = v - \frac{\langle u|v\rangle}{\langle u|u\rangle}|u\rangle$. We calculate $\langle u|v\rangle = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 5/12$, and $\langle u|u\rangle = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1/3$. Hence $w = v - 5/4u = t^2 - 1.25t + 0.25$.

4. For basis $B = \{u, v\}$, calculate two bases for $V^\star$ by specifying their action on each element of V. (1) the dual basis $\{\phi_1, \phi_2\}$, (2) the bra basis $\{\langle u|, \langle v|\}$.

   We have $x(t) = at^2 + bt - (a+b) = au + bv$. Hence $[x]_B = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, and $\phi_1(x) = a, \phi_2(x) = b$. $\langle u|x\rangle = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} -a-b \\ b \\ a \end{pmatrix} = \frac{5a+4b}{12}$. $\langle v|x\rangle = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} -a-b \\ b \\ a \end{pmatrix} = \frac{8a}{15} + \frac{5b}{12}$.