Math 524 Exam 6 Solutions

The first three problems all concern $A = \begin{pmatrix} -1/3 & -1/6 \\ 1/3 & -5/6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1/2 & 0 \\ 0 & -2/3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

1. Solve the discrete-time system given by $x(n) = Ax(n-1)$, with initial condition $x(0) = \binom{0}{1}$.

   A basis of eigenvectors is $B = \{b_1, b_2\}$, for $b_1 = (1,1)^T, b_2 = (1,2)^T$. We have $[x(0)]_B = \binom{-1}{1}$. We have $x(n) = A^n x(0) = (PDP^{-1})^n x(0) = PD^nP^{-1}x(0)$. Because $P^{-1}x = [x]_B$, we have $[x(n)]_B = D^n[x(0)]_B = D^n \binom{-1}{1} = \begin{pmatrix} -(-1/2)^n \\ (-2/3)^n \end{pmatrix}$. Hence $x(n) = P \begin{pmatrix} -(-1/2)^n \\ (-2/3)^n \end{pmatrix} = \begin{pmatrix} -(-1/2)^n + (-2/3)^n \\ -(-1/2)^n + 2(-2/3)^n \end{pmatrix}$, or $x_1(n) = (-2/3)^n - (-2)^{-n}, x_2(n) = 2(-2/3)^n - (-2)^{-n}$. One may check this with some values such as $n = 0, 1, 2$; or, one may check that this satisfies the difference equation and initial condition.

2. Solve the first-order system given by $\frac{d}{dt}x = Ax$, with initial condition given by $x(0) = \binom{0}{1}$.

   We calculate a basis of eigenvectors as before, and note that $x(0) = -b_1 + b_2$. This time $x(t) = e^{At}x(0) = e^{At}(-b_1 + b_2)$. We have $e^{At}b_1 = e^{-t/2}b_1$ and $e^{At}b_2 = e^{-2t/3}b_2$, since they are eigenvectors of A. Hence $x(t) = -e^{-t/2}b_1 + e^{-2t/3}b_2 = \begin{pmatrix} -e^{-t/2} + e^{-2t/3} \\ -e^{-t/2} + 2e^{-2t/3} \end{pmatrix}$. One may check that this satisfies the DE and initial condition.

3. Solve the second-order system given by $\frac{d^2}{dt^2}x = Ax$, with initial conditions given by $x(0) = \binom{0}{1}$ and $\dot{x}(0) = \binom{1}{0}$.

   In the B basis, we have $[x(0)]_B = \binom{-1}{1}$, $[\dot{x}(0)]_B = \binom{2}{-1}$, and $\frac{d^2}{dt^2}[x]_B = [A]_B[x]_B = D[x]_B$. For convenience, set $\omega_1 = \sqrt{1/2}$ and $\omega_2 = \sqrt{2/3}$. We can write down the solution $[x(t)]_B = \begin{pmatrix} (-1)\cos(\omega_1 t) + (2/\omega_1)\sin(\omega_1 t) \\ (1)\cos(\omega_2 t) + (-1/\omega_2)\sin(\omega_2 t) \end{pmatrix}$. We then compute $x(t) = P[x(t)]_B = \begin{pmatrix} (-1)\cos(\omega_1 t) + (2/\omega_1)\sin(\omega_1 t) + (1)\cos(\omega_2 t) + (-1/\omega_2)\sin(\omega_2 t) \\ (-1)\cos(\omega_1 t) + (2/\omega_1)\sin(\omega_1 t) + (2)\cos(\omega_2 t) + (-2/\omega_2)\sin(\omega_2 t) \end{pmatrix}$. One may check that this satisfies the initial condition (and, time permitting, the DE).

The last problem concerns $A = \begin{pmatrix} -2 & 1/2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 \\ -1/2 & 1/4 \end{pmatrix}$.

4. Solve the first-order system given by $\frac{d}{dt}x = Ax$, with initial condition given by $x(0) = \binom{0}{1}$.

   We have a basis of generalized eigenvectors $S = \{s_1, s_2\}$, for $s_1 = (1,2)^T$ (of order 1, hence an eigenvector), and $s_2 = (-1,2)^T$ (of order 2). Note that $x(0) = 1/4 s_1 + 1/4 s_2$. We have $x(t) = e^{At}x(0) = e^{At}(1/4 s_1 + 1/4 s_2)$. Because s_1 is an eigenvector, $e^{At}s_1 = e^{-t}s_1$. Because s_2 is a generalized eigenvector of order 2, $e^{At}s_2 = e^{-t}(I + (A - \lambda I)t)s_2 = e^{-t}(I + (A+I)t)s_2 = e^{-t}\begin{pmatrix} 1-t & t/2 \\ -2t & 1+t \end{pmatrix}s_2 = e^{-t}\begin{pmatrix} t-1+t \\ 2t+2(1+t) \end{pmatrix} = \begin{pmatrix} e^{-t}(2t-1) \\ e^{-t}(4t+2) \end{pmatrix}$. Putting it all together, we get $x(t) = 1/4 e^{-t}s_1 + 1/4\begin{pmatrix} e^{-t}(2t-1) \\ e^{-t}(4t+2) \end{pmatrix} = \frac{e^{-t}}{4}\begin{pmatrix} 2t \\ 4t+4 \end{pmatrix} = \begin{pmatrix} te^{-t}/2 \\ (t+1)e^{-t} \end{pmatrix}$. One may check that this satisfies the DE and initial condition.