Math 524 Exam 5 Solutions

1. Suppose that A, B are square, diagonalizable matrices satisfying $AB = BA + I$. Without using Thm. 4.10, prove that they are not simultaneously diagonalizable. (Note: Thm 4.10 says that A, B commute if and only if they are simultaneously diagonalizable).

   Suppose otherwise. Then there is some invertible P such that $PAP^{-1} = D_1, PBP^{-1} = D_2$, for some diagonal matrices D_1, D_2. We multiply by P on the left, and P^{-1} on the right, to get: $PAP^{-1}PBP^{-1} = PBP^{-1}PAP^{-1} + PIP^{-1}$, hence $D_1D_2 = D_2D_1 + I$. But diagonal matrices commute, so we subtract $D_1D_2 = D_2D_1$ from both sides to get $0 = I$, a contradiction.

The remaining problems all concern the matrix $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$.

2. Find all eigenvalues of A; give a basis for each eigenspace. HINT: each column sums to 2.

   The hint tells us that $\lambda = 2$ is one eigenvalue. The determinant is 8, the trace is 6, hence the other eigenvalues multiply to 4 $(= 8/2)$ and add to 4 $(= 6 - 2)$; we conclude that $\lambda = 2$ is the only eigenvalue, with (algebraic) multiplicity 3. We calculate $A - 2I = B = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore x_2, x_3 are free (2-dimensional eigenspace), and $x_1 = -x_2 - x_3$. One basis for E_2 is $\{(-1, 1, 0)^T, (-1, 0, 1)^T\}$.

3. Find a basis for $\mathbb{R}^3$ consisting of "power vectors" (generalized eigenvectors) of A.

   Since 2 is the only eigenvalue, we will expect $\tilde{E}_2$ to be 3-dimensional. We already have 2 eigenvectors, in the nullspace of B^1 (hence of first order). $B^2 = 0$, hence we can choose any other vector to fill out $\tilde{E}_2$, so long as it's independent with the first two we got before. This will be a generalized eigenvector of second order. For example $\{(-1, 1, 0)^T, (-1, 0, 1)^T, (1, 0, 0)^T\}$ is such a basis.

4. Write A in Jordan canonical form. You need not find the corresponding change-of-basis matrix.

   Since $m_a(2) = 3$, the JCF will have 2 in all three diagonal entries. Since $m_g(2) = 2$, there will be two blocks. Hence there are two possible answers: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

5. Evaluate $e^{At}\bar{u}$, for each $\bar{u}$ in the basis you gave in question 3. above.

   We have $e^{At} = e^{2It+Bt} = e^{2t}e^{Bt} = e^{2t}(I + Bt + (Bt)^2/2 + (Bt)^3/6 + \cdots)$. But already $B^2 = 0$, so really $e^{At} = e^{2t}(I + Bt)$. When applying this to the three vectors, note that the first two are in the nullspace of B, so we need only calculate e^{2t} instead (i.e. $e^{2t}(I + Bt)u = e^{2t}Iu + e^{2t}tBu = e^{2t}u + 0 = e^{2t}u$).

   $e^{At}(-1, 1, 0)^T = e^{2t}(-1, 1, 0)^T = (-e^{2t}, e^{2t}, 0)^T$.
   $e^{At}(-1, 0, 1)^T = e^{2t}(-1, 0, 1)^T = (-e^{2t}, 0, e^{2t})^T$.
   $e^{At}(1, 0, 0)^T = e^{2t}(I + Bt)(1, 0, 0)^T = e^{2t} \begin{pmatrix} 1-t & -t & -t \\ -t & 1-t & -t \\ 2t & 2t & 1+2t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t}(1-t) \\ e^{2t}(-t) \\ e^{2t}(2t) \end{pmatrix}$.