Math 522 Exam 9 Solutions

Theorem 1. Let $m, n \in \mathbb{N}$. If $\gcd(m,n) = 1$ then $\phi(mn) = \phi(m)\phi(n)$.
Theorem 2. Let $p, k \in \mathbb{N}$. If p is prime, then $\phi(p^k) = p^k - p^{k-1}$.

1. Use the two theorems above to prove the following:
   Claim. For all $n \in \mathbb{N}$, $\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$.

   Let $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, the unique factorization into prime powers guaranteed by the Fundamental Theorem of Arithmetic. Proof proceeds by induction on r.

   $r = 0$: Then $n = \phi(n) = 1$, and the product is empty (hence equal to 1), so the RHS is 1. Maybe you don't like this, so let's do one more base case.

   $r = 1$: Then $n = p_1^{k_1}$. By Theorem 2, $\phi(n) = p_1^{k_1} - p_1^{k_1-1} = p_1^{k_1}(1 - \frac{1}{p_1}) = n(1 - \frac{1}{p_1})$, as desired.

   $r > 1$: Write $n = (p_1^{k_1})m$, where $m = p_2^{k_2}\cdots p_r^{k_r}$. Applying both theorems, we get $\phi(n) = (p_1^{k_1} - p_1^{k_1-1})\phi(m)$. Applying the inductive hypothesis, we get $\phi(n) = p_1^{k_1}(1 - \frac{1}{p_1})m\prod_{p|m}(1 - \frac{1}{p}) = n(1 - \frac{1}{p_1})\prod_{p|m}(1 - \frac{1}{p})$=RHS, as desired.

2. Compute $\phi(150), d(150)$, and $\sigma(150)$.

   We begin by factoring $150 = 2 \cdot 3 \cdot 5^2$.
   $\phi(150) = 150\prod_{p|150}(1 - \frac{1}{p}) = 150(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 40$.
   $d(150) = (1 + 1)(1 + 1)(1 + 1 + 1) = 2 \cdot 2 \cdot 3 = 12$.
   $\sigma(150) = (1 + 2)(1 + 3)(1 + 5 + 25) = 3 \cdot 4 \cdot 31 = 372$.