Math 522 Exam 8 Solutions

1. For $f(x) = 47x^2 + x - 2$, find all solutions to $f(x) \equiv 0 \pmod{47^2}$.
   BONUS: Find all solutions to $f(x) \equiv 0 \pmod{47^3}$.

   We use the lifting theorem, so we first solve $f(x) \equiv 0 \pmod{47}$. Conveniently, $47x^2 + x - 2 \equiv x - 2 \pmod{47}$, so $x = 2$ is the unique root modulo 47. Now, $f'(x) = 94x + 1$, so $f'(2) = 189 \equiv 1 \pmod{47}$, so this root will lift to a unique root modulo 47^2. We solve $f'(2)t \equiv -f(2)/47 \pmod{47}$, which simplifies to $1t \equiv -\frac{188}{47} = -4 \equiv 43 \pmod{47}$. Hence $x = 2 + 43 \cdot 47 = 2023$ is the unique root of $f(x)$ modulo $47^2 = 2209$.

   BONUS: We start with the sole root $r = 2023$, modulo 47^2. We have $f'(2023) = 94 \cdot 2023 + 1 \equiv 1 \pmod{47}$, so again this root will lift uniquely modulo 47^3. We solve $f'(2023)t \equiv -f(2023)/47^2 \pmod{47}$, which simplifies to $1t \equiv -87076 \equiv 15 \pmod{47}$. Hence $x = 2023 + 15 \cdot 47^2 = 35158$ is the unique root of $f(x)$ modulo $47^3 = 103823$.

2. For $n \in \mathbb{N}$, prove that $\phi(n)$ is even if and only if $n > 2$.

   Suppose that $p^a | n$ for any odd prime p and $a \in \mathbb{N}$, then (since ϕ is multiplicative) we take a maximal and have $\phi(n) = \phi(p^a)\phi(\frac{n}{p^a}) = (p^a - p^{a-1})\phi(\frac{n}{p^a})$. But p^a is odd, and so is p^{a-1}, so their difference is even, and so hence is $\phi(n)$. Hence $\phi(n)$ is even for every n that is not a power of 2 (powers of 2 have not yet been addressed). Now $\phi(2^a) = 2^a - 2^{a-1}$. This is even for $a \geq 2$, being the difference of two even numbers. Hence $\phi(n)$ is even for every n except possibly $n = 1, 2$. But in fact $\phi(1) = \phi(2) = 1$, which are odd.

3. High score=101, Median score=77, Low score=50