Math 522 Exam 3 Solutions

1. In the country of Vadimia, license plates all contain exactly five characters. The first three must be letters (A-Z), but the last two two may be either letters (A-Z) or numbers (0-9). No letter may appear twice; numbers may be repeated however. How many license plates are possible?

   Let S denote the set of all valid licence plates. We partition $S = S_1 \cup S_2 \cup S_3 \cup S_4$ as follows. S_1 consists of all letters; S_2 has the last character is a number; S_3 has the next-to-last character a number; S_4 has the last two characters numbers. We have $|S_1| = {}_{26}P_5$, $|S_2| = |S_3| = {}_{26}P_4 \cdot 10$, $|S_4| = {}_{26}P_3 \cdot 10 \cdot 10$. Hence $|S| = 7,893,600 + 3,588,000 + 3,588,000 + 1,560,000 = 16,629,600$.

   Alternate solution: $S = S_1 \cup S_2$, where S_1 has a letter in the 4th place, and S_2 has a number in the 4th place. $|S_1| = 26 \cdot 25 \cdot 24 \cdot 23 \cdot (22 + 10) = 11,481,600$. $|S_2| = 26 \cdot 25 \cdot 24 \cdot 10 \cdot (23 + 10) = 5,148,000$.

2. Let $E = \{2k : k \in \mathbb{Z}\}$, the set of even integers, under the multiplication operation. We call $m \in E$ *irreducible* if there are no $a, b \in E$ with $m = ab$.

   (a) For every odd integer k, prove that $2k$ is irreducible.

   Let $2k = ab$. Treating $2k$ as an integer, we consider its prime factorization (unique by Thm 2-5). 2 appears exactly once. Hence, 2 appears exactly once in the integer ab; therefore either a or b is not a multiple of 2 and hence not in E.

   (b) Find two different factorizations of $2^2 \cdot 3 \cdot 5 \cdot 7 (= 420)$ into irreducibles.

   $420 = (2 \cdot 3)(2 \cdot 5 \cdot 7) = (2 \cdot 5)(2 \cdot 3 \cdot 7)$. *Each of 6,70,10,42 are irreducible by (a).*

   (c) For every odd integer $k > 1$, set $x = 2k, y = 2k^2, z = 2$. Prove that $x|yz$ but $x \nmid y$ and $x \nmid z$. (Recall that $a|b$ means that $b/a \in E$)

   $yz = 4k^2 = x \cdot x$, hence $x|yz$. $y/x = k$, which is odd and hence not in E; hence $x \nmid y$. $z/x = 1/k$, which is in $(0, 1)$ and hence not in $\mathbb{Z}$, much less in E; hence $x \nmid z$.

   We may easily show that the only irreducibles of E are those in (a). We have $2|6 \cdot 6$, since $36 = 2 \cdot 18$, but $2 \nmid 6$. These two facts, together with (c), show that every irreducible in E is not prime. Hence, E has NO primes, since every prime must be irreducible. (Recall that x is a prime when $x|yz$ implies $x|y$ or $x|z$). Note also that E is not a ring, since it does not contain 1. Some mathematicians (in the minority) still call this a ring, while others use terms as pseudo-ring *or* rng.

   BONUS: Improve on (b), if possible: find the smallest positive integer in E that has at least two different factorizations into irreducibles.

   Each irreducible must have exactly one 2 in its prime factorization. Further, we need at least three different irreducibles; hence there must be at least two more primes. The smallest two primes are 3,3; hence the smallest integer in E with more than one factorization is $2^2 3^2 = 36 = 6 \cdot 6 = 2 \cdot 18$. Each of 2,6,18 are irreducible by (a).

3. Exam grades: 105, 101, 100, 95, 95, 93, 93, 86, 82, 82, 78, 74, 69, 61