Math 522 Exam 2 Solutions

1. Find all pairs of positive integers (x, y) such that $21x + 15y = 363$, if any exist.

   We find $3 = \gcd(21, 15)$*, and* $3|363$*, so integer solutions exist (though not necessarily positive integer solutions). We turn to our old favorite, trial and error.* $x = 1$ *gives* $y = {}^{114}/_{5}$*,* $x = 2$ *gives* $y = {}^{107}/_{5}$*, but* $x = 3$ *gives the solution* $(3, 20)$*. To find other solutions, add* $5 = {}^{15}/_{3}$ *to* x *and subtract* $7 = {}^{21}/_{3}$ *from* y*, to get the other two solutions* $(8, 13)$ *and* $(13, 6)$*. No other positive integer solutions exist, since all other integer solutions have either* x *or* y *negative.*

2. Let m, n, a, b be integers, with $b \neq 0$. Suppose that $\gcd(m, ab) = 1$. Prove that $\gcd(ma + nb, b) = \gcd(a, b)$.

   Note that $\gcd(m, a) \in CD(m, ab)$*, so* $\gcd(m, a) | \gcd(m, ab) = 1$*. Similarly,* $\gcd(m, b) = 1$*. By part 5 of the GCD theorem proved in class, since* $\gcd(m, a) = 1$*, we must have* $\gcd(m, b)\gcd(a, b) = \gcd(am, b)$*. However, since* $\gcd(m, b) = 1$*, we must in fact have* $\gcd(a, b) = \gcd(am, b)$*. By part 3 of the same theorem, we have* $\gcd(am, b) = \gcd(am + nb, b)$*. Combining these two gives* $\gcd(a, b) = \gcd(am + nb, b)$*, as desired.*

   Alternate solution: Let $d = \gcd(ma, b), e = \gcd(a, b)$*. First, since* $e \in CD(ma, b)$ *we have* $e \leq d$*. Second, we multiply the two equations* $\alpha m + \beta ab = 1, \gamma a + \delta b = e$*, to get* $am(\alpha\gamma) + b(\beta\gamma a^2 + \beta\delta ab + \alpha\delta m) = e$*. Hence* $e \in PS(ma, b)$ *so* $e \geq d$*. Hence* $e = d$*.*

3. Exam grades: 98, 90, 90, 88, 85, 83, 81, 80, 80, 77, 75, 75, 75, 73, 70