MATH 521B: Abstract Algebra

Homework 8: Due Mar. 23

1. Set $H = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : a, b, c, d \in \mathbb{R}, ad - bc > 0 \}$. Determine $GL(2, \mathbb{R})/H$, and find some familiar group it is isomorphic to.

2. Determine $GL(2, \mathbb{R})/SL(2, \mathbb{R})$, and find some familiar group it is isomorphic to.

3. Set $G = \mathbb{Z} \times \mathbb{Z}$, where the operation is addition. Let $S = \langle (5,0), (0,5) \rangle$. Prove that $G/S \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

4. Set $G = \mathbb{Z} \times \mathbb{Z}$, where the operation is addition. Let $N = \langle (5,5) \rangle$. Prove that $G/N \cong \mathbb{Z} \times \mathbb{Z}_5$.

5. Set $G = \mathbb{Z} \times \mathbb{Z}$, where the operation is addition. Set $M = \{(x, -x) : x \in \mathbb{Z}\}$, a subset of G. Prove that $M \trianglelefteq G$, and that $G/M \cong \mathbb{Z}$.

6. Suppose $f : G \to H$ is a homomorphism. Prove that f is one-to-one if and only if $|Ker(f)| = 1$.

For problems 7-9, let G, H be groups, and consider $G^\star = \{(a, id) : a \in G\}$, a subgroup of $G \times H$.

7. Prove that $G^\star \cong G$.

8. Prove that $G^\star \trianglelefteq G \times H$.

9. Prove that $(G \times H)/G^\star \cong H$.

For problems 10-13, for any group G we set $S = \{xyx^{-1}y^{-1} : x, y \in G\}$ and define the commutator subgroup $G' = \langle S \rangle$, the subgroup of G generated by all the elements of S.

10. Prove that $G' \trianglelefteq G$.

11. Prove that G/G' is abelian.

12. If $N \trianglelefteq G$ and G/N is abelian, prove that $G' \subseteq N$.

13. If $M \leq G$ and $G' \subseteq M$, prove that $M \trianglelefteq G$.