MATH 521B: Abstract Algebra
Exam 2 Solutions

For the first four problems we fix $G \leq SL(2,\mathbb{R})$, defined as $G = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a,b,d \in \mathbb{R}, ad=1\}$, and $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{R}\}$.

1. Prove that $N \trianglelefteq G$.

Let $x = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G$, and $n = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in N$. We calculate $xnx^{-1} = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} ad & -ab+a^2t+ab \\ 0 & da \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a^2t \\ 0 & 1 \end{smallmatrix}\right) \in N$. Since $n \in N$ was arbitrary, we conclude that $xNx^{-1} \subseteq N$. Since $x \in G$ was arbitrary, we can apply our normal theorem to get $N \trianglelefteq G$.

2. For each $x \in G$, determine explicitly its equivalence class $[x]$, modulo N.

Fix $x = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G$. We have $[x] = \{y \in G : xy^{-1} \in N\}$. For $y = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$, we calculate $xy^{-1} = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} d' & -b' \\ 0 & a' \end{smallmatrix}\right) = \left(\begin{smallmatrix} ad' & -ab'+ba' \\ 0 & da' \end{smallmatrix}\right)$. This is in N exactly when $ad' = 1 = da'$, i.e. $a = a', d = d'$. Hence we can write explicitly $[x] = \{\left(\begin{smallmatrix} a & b' \\ 0 & d \end{smallmatrix}\right) : b' \in \mathbb{R}\}$.

3. Prove that $N \cong \mathbb{R}$.

We explicitly write down the isomorphism $f : N \to \mathbb{R}$ via $f : \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mapsto b$. We check:
Homomorphism: $f(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right)) = f(\left(\begin{smallmatrix} 1 & b+b' \\ 0 & 1 \end{smallmatrix}\right)) = b + b' = f(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)) + f(\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right))$.
One-to-one: Suppose $f(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)) = f(\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right))$. Then $b = b'$, so $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right)$.
Onto: Let $b \in \mathbb{R}$. Then $f(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)) = b$.

4. Prove that $G/N \cong \mathbb{R}^\times$.

We write down $f : G \to \mathbb{R}^\times$ via $f : \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto a$. We want to apply FIT, so we check:
Homomorphism: $f(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)) = f(\left(\begin{smallmatrix} aa' & ab'+bd' \\ 0 & dd' \end{smallmatrix}\right)) = aa' = f(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))f(\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right))$.
Onto: Let $a \in \mathbb{R}^\times$. Then $f(\left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right)) = a$.
We calculate $Ker(f) = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G : f(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)) = 1\} = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G : a = 1\} = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{R}\} = N$. We now apply the First Isomorphism Theorem and are done.

5. Fix a group G, with $N \trianglelefteq G$. Suppose that $[G : N] = 20$. Prove that $a^{20} \in N$, for all $a \in G$.

Let $a \in G$. Consider $Na \in G/N$. A corollary to Lagrange's theorem tells us $h^{|H|} = id$, for any h in any group H. Hence $Nid = (Na)^{|G/N|} = (Na)^{[G:N]} = N(a^{[G:N]}) = Na^{20}$. Thus $[a^{20}] = [id]$, and we have proved this implies $a^{20}(id)^{-1} \in N$, so $a^{20} \in N$.

6. Fix abelian group G, with $|G| = 2k$, and k odd. Prove that G has exactly one element g with $|g| = 2$.

We first prove there is at least one such element (this was on the Exam 1 Prep). Pair each element of G with its inverse. The identity is alone in a pair, being its own inverse, as are all elements of order 2, but nothing else. There are $2k$ elements altogether, so there must be at least one non-identity element, with order 2.

Suppose now that g, h each have order 2. Set $K = \langle g, h \rangle = \{id, g, h, gh\}$. Since G is abelian in fact $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Lagrange's theorem $|K| = 4$ must divide $|G|$, but this is a contradiction.

7. Fix groups G, H, and suppose $A \trianglelefteq G$ and $B \trianglelefteq H$. Prove that $(A \times B) \trianglelefteq (G \times H)$.

We first prove that $(A \times B) \leq (G \times H)$, by observing that $(a, b)(a', b') = (aa', bb') \in A \times B$, and $(a, b)^{-1} = (a^{-1}, b^{-1}) \in A \times B$.

Now, let $(g, h) \in G \times H$. We have $(g, h)(a, b)(g, h)^{-1} = (g, h)(a, b)(g^{-1}, h^{-1}) = (gag^{-1}, hbh^{-1})$. Since $A \trianglelefteq G$ we have $gAg^{-1} \subseteq A$, so there is some $a' \in A$ with $gag^{-1} = a'$. Similarly there is some $b' \in B$ with $hbh^{-1} = b'$. Hence $(gag^{-1}, hbh^{-1}) = (a', b') \in A \times B$. Since $(a, b) \in A \times B$ was arbitrary, in fact $(g, h)(A \times B)(g, h)^{-1} \subseteq A \times B$. Since $(g, h) \in G \times H$ was arbitrary, by our normal theorem $(A \times B) \trianglelefteq (G \times H)$.

8. Fix a group G. Set $A = \{K : |K| = 20,\ K \leq G\}$, the set of all subgroups of order 20, and assume that A is nonempty. Set $N = \bigcap_{K \in A} K$. Prove that $N \trianglelefteq G$, assuming $N \leq G$.

Let $g \in G$, $n \in N$, $K \in A$, and consider gng^{-1}. The difficult part of this problem is to prove that a generic element, gng^{-1}, is in a generic (and hence every) K.

Set $t = |g|$, and define $L = g^{t-1}K(g^{t-1})^{-1} = g^{t-1}Kg^{1-t}$. We proved in Homework 7 that, since L is a conjugate of K, they have the same order; i.e. $|L| = |K| = 20$. Hence $L \in A$, and since $n \in \bigcap A$, in fact $n \in L$. Hence there is some $k \in K$ such that $n = g^{t-1}kg^{1-t}$. Plugging in, we get $gng^{-1} = gg^{t-1}kg^{1-t}g^{-1} = g^t k g^{-t} = k \in K$.

Since $K \in A$ was arbitrary, gng^{-1} is in all $K \in A$. Hence $gng^{-1} \in N$. Since $n \in N$ was arbitrary, $gNg^{-1} \subseteq N$. Since $g \in G$ was arbitrary, we apply our normal theorem to conclude that $N \trianglelefteq G$.