MATH 521A: Abstract Algebra

Homework 7: Due Nov. 1

1. Consider the ring $\mathbb{Z}_4[x]$. Prove that $x + 2x^k$ divides x^3, for every $k \in \mathbb{N}$.
   [This is one reason why we like to restrict to $F[x]$ rather than $R[x]$.]

2. Find a monic associate of $(1 + 2i)x^3 + x - 1$ in $\mathbb{C}[x]$.

3. For each $a \in \mathbb{Z}_7$, factor $x^2 + ax + 1$ into irreducibles in $\mathbb{Z}_7[x]$.

4. For each $a, b \in \mathbb{Z}_3$, factor $x^2 + ax + b$ into irreducibles in $\mathbb{Z}_3[x]$.

5. Find some $f(x) \in \mathbb{Z}_5[x]$ that is monic, of degree 4, reducible, but with no roots.

6. Factor $x^7 - x$ as a product of irreducibles in $\mathbb{Z}_7[x]$.

7. Let $a, b \in \mathbb{N}$ be distinct, and each greater than 1. Set $n = ab$. Find a quadratic polynomial in $\mathbb{Z}_n[x]$ with at least three distinct roots.

8. Let $a, b, c \in F$ with $a \neq 0$. Set $f(x) = ax^2 + bx + c$. Suppose that $r, s \in F$ are distinct roots of $f(x)$. Prove that $r + s = -a^{-1}b$ and that $rs = a^{-1}c$.

9. Let $a \in F$ and define $\tau_a : F[x] \to F$ via $\tau_a : f(x) \mapsto f(a)$. Prove that τ_a is a surjective (ring) homomorphism, but not an isomorphism.

10. Set $f(x) = x^6 + 2x^4 + 3x^3 + 1$. Find some prime p such that $x - 2$ is a divisor of $f(x)$ in $\mathbb{Z}_p[x]$. Then factor $f(x)$ into irreducibles in $\mathbb{Z}_p[x]$.