MATH 521A: Abstract Algebra
Homework 10: Due Dec. 6

1. Prove that $T = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. Note that $\mathbb{Q}$ is a subfield of T.

2. Let F, G be rings such that $\mathbb{Q}$ is a subring of each. Suppose $f : F \to G$ is a (ring) isomorphism. Prove that, for every $a \in \mathbb{Q}$, in fact $f(a) = a$.

3. Prove that $R = \mathbb{Q}[x]/(x^2 - 2)$ is not isomorphic to $S = \mathbb{Q}[x]/(x^2 - 3)$. Hint: problem 2.

4. Prove that $R = \mathbb{Q}[x]/(x^2 - 2)$ is isomorphic to $S = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

5. Set $F = \mathbb{Z}_3[x]/(x^3 - x + 1)$. Prove that $f(x) = x^3 - x + 1$ splits in F. That is, find three distinct roots of $f(x)$ in F.

6. Prove that $\{1, \sqrt{2}, i, i\sqrt{2}\}$ is linearly independent over $\mathbb{Q}$.

7. Set $R = \mathbb{Q}(\sqrt{2})$, and $S = R(i)$. Determine $[R : \mathbb{Q}]$, $[S : R]$, and $[S : \mathbb{Q}]$.

8. Prove that $x^4 - 2x^2 + 9$ is the minimal polynomial for $i + \sqrt{2}$ over $\mathbb{Q}$. (remember to prove irreducibility)

9. Set $T = \mathbb{Q}(i + \sqrt{2})$, and let R, S be as in problem 7. Prove that $1, \sqrt{2}, i, i\sqrt{2}$ are all in T, so $S \subseteq T$.

10. Let R, S, T be as in problems 7 and 9. Determine $[T : \mathbb{Q}]$, and hence $[T : S]$. What can we conclude about S, T?