MATH 521A: Abstract Algebra

Homework 3 Solutions

1. Write the $\oplus$-addition and $\odot$-multiplication tables of $\mathbb{Z}_{10}$.

$\oplus$	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[7]	[8]	[9]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[7]	[8]	[9]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[7]	[8]	[9]	[0]	[1]	[2]	[3]	[4]	[5]
[7]	[7]	[8]	[9]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[8]	[8]	[9]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[9]	[9]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

$\odot$	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
[2]	[0]	[2]	[4]	[6]	[8]	[0]	[2]	[4]	[6]	[8]
[3]	[0]	[3]	[6]	[9]	[2]	[5]	[8]	[1]	[4]	[7]
[4]	[0]	[4]	[8]	[2]	[6]	[0]	[4]	[8]	[2]	[6]
[5]	[0]	[5]	[0]	[5]	[0]	[5]	[0]	[5]	[0]	[5]
[6]	[0]	[6]	[2]	[8]	[4]	[0]	[6]	[2]	[8]	[4]
[7]	[0]	[7]	[4]	[1]	[8]	[5]	[2]	[9]	[6]	[3]
[8]	[0]	[8]	[6]	[4]	[2]	[0]	[8]	[6]	[4]	[2]
[9]	[0]	[9]	[8]	[7]	[6]	[5]	[4]	[3]	[2]	[1]

2. For $\mathbb{Z}_{10}$, find the neutral additive element, the neutral multiplicative element, and all zero divisors.

The neutral additive and multiplicative elements are [0] and [1]. The zero divisors are [2],[4],[5],[6],[8].

3. Find the units of $\mathbb{Z}_{10}$; for each unit specify its inverse.

The units are exactly those nonzero elements that are not zero divisors. They are [1] (its own inverse),[3] (inverse of [7]),[7] (inverse of [3]), and [9] (its own inverse).

4. The *additive order* of an element in $\mathbb{Z}_{10}$ is the number of times one must $\oplus$-add it to itself to get [0]. Determine the additive order of each element of $\mathbb{Z}_{10}$.

[0] has order 1; [5] has order 2; [2],[4],[6];[8] have order 5; and [1],[3],[7],[9] all have order 10.

We define $\mathbb{Z}_2 \times \mathbb{Z}_5 = \{(a, b) : a \in \mathbb{Z}_2, b \in \mathbb{Z}_5\}$, the set of ordered pairs of elements, one each from $\mathbb{Z}_2$ and $\mathbb{Z}_5$. We define operations in the natural way, i.e. componentwise:
$(a, b) \oplus (a', b') = (a \oplus_2 a', b \oplus_5 b')$ and $(a, b) \odot (a', b') = (a \odot_2 a', b \odot_5 b')$.

5. Write the $\oplus$-addition and $\odot$-multiplication tables of $\mathbb{Z}_2 \times \mathbb{Z}_5$.

$\oplus$	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])
([0], [0])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])
([1], [1])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])	([0], [0])
([0], [2])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])	([0], [0])	([1], [1])
([1], [3])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])	([0], [0])	([1], [1])	([0], [2])
([0], [4])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])	([0], [0])	([1], [1])	([0], [2])	([1], [3])
([1], [0])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])
([0], [1])	([0], [1])	([1], [2])	([0], [3])	([1], [4])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])
([1], [2])	([1], [2])	([0], [3])	([1], [4])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])
([0], [3])	([0], [3])	([1], [4])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])
([1], [4])	([1], [4])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])

$\odot$	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])
([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])	([0], [0])
([1], [1])	([0], [0])	([1], [1])	([0], [2])	([1], [3])	([0], [4])	([1], [0])	([0], [1])	([1], [2])	([0], [3])	([1], [4])
([0], [2])	([0], [0])	([0], [2])	([0], [4])	([0], [1])	([0], [3])	([0], [0])	([0], [2])	([0], [4])	([0], [1])	([0], [3])
([1], [3])	([0], [0])	([1], [3])	([0], [1])	([1], [4])	([0], [2])	([1], [0])	([0], [3])	([1], [1])	([0], [4])	([1], [2])
([0], [4])	([0], [0])	([0], [4])	([0], [3])	([0], [2])	([0], [1])	([0], [0])	([0], [4])	([0], [3])	([0], [2])	([0], [1])
([1], [0])	([0], [0])	([1], [0])	([0], [0])	([1], [0])	([0], [0])	([1], [0])	([0], [0])	([1], [0])	([0], [0])	([1], [0])
([0], [1])	([0], [0])	([0], [1])	([0], [2])	([0], [3])	([0], [4])	([0], [0])	([0], [1])	([0], [2])	([0], [3])	([0], [4])
([1], [2])	([0], [0])	([1], [2])	([0], [4])	([1], [1])	([0], [3])	([1], [0])	([0], [2])	([1], [4])	([0], [1])	([1], [3])
([0], [3])	([0], [0])	([0], [3])	([0], [1])	([0], [4])	([0], [2])	([0], [0])	([0], [3])	([0], [1])	([0], [4])	([0], [2])
([1], [4])	([0], [0])	([1], [4])	([0], [3])	([1], [2])	([0], [1])	([1], [0])	([0], [4])	([1], [3])	([0], [2])	([1], [1])

6. For $\mathbb{Z}_2 \times \mathbb{Z}_5$, find the neutral additive element, the neutral multiplicative element, and all zero divisors.

   The neutral additive and multiplicative elements are $([0],[0])$ and $([1],[1])$. The zero divisors are $([0],[1])$, $([0],[2])$, $([0],[3])$, $([0],[4])$, $([1],[0])$.

7. Find the units of $\mathbb{Z}_2 \times \mathbb{Z}_5$; for each unit specify its inverse.

   The units are exactly those nonzero elements that are not zero divisors. They are $([1],[1])$ (its own inverse), $([1],[3])$ (inverse of $([1],[2])$), $([1],[2])$ (inverse of $([1],[3])$), and $([1],[4])$ (its own inverse).

8. Determine the additive order of each element of $\mathbb{Z}_2 \times \mathbb{Z}_5$.

   $([0],[0])$ has order 1; $([1],[0])$ has order 2; $([0],[1])$, $([0],[2])$, $([0],[3])$, $([0],[4])$ have order 5; and $([1],[1])$, $([1],[2])$, $([1],[3])$, $([1],[4])$ all have order 10.

9. Compare the two rings $\mathbb{Z}_{10}$ and $\mathbb{Z}_2 \times \mathbb{Z}_5$ as best you can.

   The two rings are "the same". Although the elements have different names, if we change the names correctly we get identical structure. The jargon for this is that these rings are "isomorphic", which we will study soon.