# MATH 521A: Abstract Algebra Final Exam

Please read the following instructions. For the following exam you are free to use a single sheet of paper with notes, but no other materials. Please turn in **exactly nine** problems. You must do problems 1-5, and four more, chosen from 6-12. Please write your answers on separate paper, make clear what work goes with which problem, and put your name or initials on every page. You have 120 minutes. Each problem will be graded on a 5-10 scale (as your quizzes), for a total score between 45 and 90. This will then be multiplied by $\frac{10}{9}$ for your exam score. $R, S$ denote rings, not necessarily commutative or with identity. $F$ denotes a field.

**Turn in problems 1,2,3,4,5:**

1. Carefully define the term $\mathbb{Z}_n$.
2. Carefully state the Rational Root Test Theorem.
3. Let $\phi : R \to S$ be a ring isomorphism. Prove that $R$ has an identity if and only if $S$ has an identity.
4. Construct a field with twenty-five elements, and list all the elements.
5. Let $R = \mathbb{Z}[x]$, $I = (x - 1)$. Prove that $I$ is prime, and not maximal.

**Turn in exactly four more problems of your choice:**

6. Prove the division algorithm in $\mathbb{Z}$.
7. Let $a \in R$. Prove that $aR = \{ar : r \in R\}$ is a subring of $R$.
8. Let $\phi : F \to R$ be a ring homomorphism. Prove that either
   (a) for all $x \in F$, $\phi(x) = 0$; or (b) $\phi$ is injective.
9. Define $R \subseteq \mathbb{Z}_2[x]$ via $R = \{f(x) : f(0) + f(1) = 0\}$. Prove or disprove that $R$ is a subring.
10. Set $f(x) = x^3 + 2x^2 + 2x + 2 \in \mathbb{Z}_4[x]$. Prove that $f(x)$ is irreducible, and not prime.
11. Find the equivalence classes and rules for addition and multiplication in $\mathbb{Q}[x]/(x^2 - 1)$. Find all the units and zero divisors.
12. Let $R = \mathbb{Z}[x]$, $p$ prime. Let $I = \{pa_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{Z}\}$. Prove that $I$ is an ideal, and maximal.