MATH 521A: Abstract Algebra
Preparation for Exam 2

1. For ring R, define its *center* $Z(R) = \{a \in R : \forall r \in R, ar = ra\}$. Prove that $Z(R)$ is a subring of R.
2. For ring R, and $x \in R$, define the *centralizer of* x $C_x(R) = \{a \in R : ax = xa\}$. Prove that $C_x(R)$ is a subring of R.
3. Let R be a ring, and S_1, S_2 both subrings of R. Prove or disprove that $S_1 \cup S_2$ must be a subring of R.
4. Let $R = \mathbb{Z}, U = 17\mathbb{Z}$, two rings. Suppose $U \subseteq V \subseteq R$, and V is a ring. Prove that $V = U$ or $V = R$.
5. Prove that $\mathbb{Q}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$ is a commutative ring with identity.
6. For ring R with $a, b \in R$, we say a is a *left divisor* of b if there is some $c \in R$ with $ac = b$. Suppose that R is a field, $a, b \in R$, and $a \neq 0$. Prove that a is a left divisor of b.
7. With left divisors defined as in problem 6, let $R = M_{2,2}(\mathbb{R})$, $a = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right), b = \left(\begin{smallmatrix} 2 & 2 \\ 0 & 2 \end{smallmatrix}\right)$. Determine whether or not a is a left divisor of b, and whether or not b is a left divisor of a.
8. For the next four problems, let $X = \{1, 2, 3, \ldots, 100\}$, and let the power set of X, denoted $\mathcal{P}(X)$, be the set of all subsets of X. Let R have ground set $\mathcal{P}(X)$, with operations $a \odot b = a \cap b$ and
$$a \oplus b = a \Delta b = (a \setminus b) \cup (b \setminus a) = (a \cup b) \setminus (a \cap b)$$
Prove that R is a commutative ring with identity.
9. For R as in problem 8, for all $a \in R$, define $\overline{a} = 1_R \oplus a$. Prove that (i) $a \odot a = a$; (ii) $a \oplus a = 0$; (iii) $\overline{a} = X \setminus a$ (the complement of a); (iv) $a \oplus \overline{a} = 1_R$; (v) $a \odot \overline{a} = 0_R$.
10. For R as in problem 8, define $f : R \to \mathbb{Z}_2$ via $f : x \mapsto \begin{cases} [0] & 7 \notin x \\ [1] & 7 \in x \end{cases}$. Prove that f is a homomorphism.
11. For $X, \mathcal{P}(X)$ as in problem 8, define S with ground set $\mathcal{P}(X)$ and operations $a \odot b = a \cap b$ and $a \boxplus b = a \cup b$. Prove that S is *not* a ring.
12. Let R be a ring such that $x^2 = 0$ for all $x \in R$. For all $a, b \in R$, prove that a commutes with $ab + ba$.
13. Suppose R has all the ring axioms except $a + b = b + a$. Prove that axiom from the others.
14. Let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Consider the function $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ given by $f : [a]_{mn} \mapsto ([a]_m, [a]_n)$, proved a homomorphism in class. Prove that f is a bijection.
15. For ring R, $x \in R$, and $n \in \mathbb{N}$, we say x has *additive order* n if $\underbrace{x + x + \cdots + x}_{n} = 0_R$, and for $m < n$ we have $\underbrace{x + x + \cdots + x}_{m} \neq 0_R$. We write this $ord_R(x) = n$. Suppose we have a homomorphism $f : R \to S$, and $x \in R$ has an additive order. Prove that $ord_S(f(x)) | ord_R(x)$, i.e. the order of $f(x)$ divides the order of x.
16. With additive order as defined in problem 15, suppose that $x \in R$ has an order and $f : R \to S$ is an isomorphism. Prove that $ord_R(x) = ord_S(f(x))$.
17. Let $R = M_{2,2}(\mathbb{Z})$ and $S = \mathbb{Z}$. Prove or disprove that trace:$R \to S$ given by $trace \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a + d$ is a homomorphism.
18. Let $R = M_{2,2}(\mathbb{Z})$ and $S = \mathbb{Z}$. Prove or disprove that det:$R \to S$ given by $det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc$ is a homomorphism.
19. Let R be the set of all continuous real-valued functions defined on $[0, 1]$, with the natural ring operations $(f + g)(x) = f(x) + g(x), (fg)(x) = f(x)g(x)$. Prove that R is a commutative ring with 1_R.
20. Let R be the ring from problem 19, and define $\phi : R \to \mathbb{R}$ as $\phi : f \mapsto f(1/2)$. Prove that ϕ is a homomorphism, and find its kernel and image.