Math 254 Spring 2014 Exam 9 Solutions

1. Carefully state the definition of the "independent". Give two examples, drawn from $\mathbb{R}^2$.

   A set of vectors is independent if no nondegenerate linear combination yields the zero vectors. Two examples are $\{(1,1)\}$ and $\{(1,1),(2,3)\}$.

2. Prove that for all square matrices A, B, if A is similar to B then B is similar to A.

   If A is similar to B then there is a square matrix such that $B = P^{-1}AP$. Multiply on the left by P to get $PB = PP^{-1}AP = AP$. Multiply on the right by P^{-1} to get $PBP^{-1} = APP^{-1} = A$. Set $Q = P^{-1}$; note that $Q^{-1} = (P^{-1})^{-1} = P$, and we have $A = Q^{-1}BQ$. Hence B is similar to A.

   The remaining three problems are all in vector space $V = Span(S)$, where $S = \{e^t, e^{-t}\}$. They all concern $F : V \to V$ given by $F = \frac{d}{dt}$.

3. Find the rank and nullity of F.

   You need any two of the following three ideas:

   Nullity: If $F(ae^t + be^{-t}) = 0$ then $ae^t - be^{-t} = 0$. We rearrange as $ae^t = be^{-t}$ and $ae^{2t} = b$. The only way for this to hold for all t is if $a = b = 0$. Hence the kernel of F is 0-dimensional and the nullity is 0.

   Rank: $F(e^t) = e^t$ and $F(e^{-t}) = -e^{-t}$. The rank of F is the dimension of $Span(X)$, for $X = \{e^t, -e^{-t}\}$. Since X is independent, the rank of F is 2.

   Rank-Nullity Theorem: The domain of F is V, which has dimension 2 (since e^t, e^{-t} are independent), so the rank of F plus the nullity of F equals 2.

4. Find $[F]_S$.

   We have $F(e^t) = e^t = 1e^t + 0e^{-t}$, and $F(e^{-t}) = -e^{-t} = 0e^t - 1e^{-t}$. Putting these as columns, we get $[F]_S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

5. Recall the hyperbolic functions $\sinh t = \frac{e^t - e^{-t}}{2}$, $\cosh t = \frac{e^t + e^{-t}}{2}$. Set $T = \{\sinh t, \cosh t\}$. Find $[F]_T$.

   Method 1: We compute derivatives directly to get $F(\sinh t) = F(\frac{1}{2}e^t - \frac{1}{2}e^{-t}) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t$, $F(\cosh t) = F(\frac{1}{2}e^t + \frac{1}{2}e^{-t}) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t$. Hence $[F]_T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

   Method 2: We compute the change-of-basis matrices $P_{ST} = \left(\begin{smallmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{smallmatrix}\right)$ and $P_{TS} = P_{ST}^{-1} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $[F]_T = P_{TS}[F]_S P_{ST} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.