Math 254 Spring 2014 Exam 5 Solutions

1. Carefully state the definition of "spanning". Give two spanning sets, drawn from $P_1(t)$.

   A set of vectors, drawn from vector space V, is spanning if its span is the entire space V. Two spanning sets for $P_1(t)$ are $\{1, t\}$ and $\{1+t, 1-t, 2+3t\}$.

2. Give the standard basis for $M_{2,2}$.

   $\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$ To receive full credit, a solution must be a set containing these four matrices, in this order.

The remaining three problems are all in $V = \mathbb{R}^4$. Set $r = (1,1,0,0), s = (0,1,1,0)$, $u = (1,1,1,1), v = (1,0,1,0), w = (0,1,0,1)$. Set $U_1 = Span(r,s),\ U_2 = Span(u,v,w)$.

3. Find a basis for U_1, and a basis for U_2.

   Two vectors are dependent if one is a multiple of the other; since neither of r, s is a multiple of the other, then $\{r, s\}$ is independent. Thus $\{r, s\}$ is a minimal spanning set for U_1, and hence a basis for U_1. This test does NOT work for more than two vectors.

   To find a basis for U_2, we row reduce $\begin{pmatrix} -u- \\ -v- \\ -w- \end{pmatrix} = \begin{pmatrix} 1&1&1&1 \\ 1&0&1&0 \\ 0&1&0&1 \end{pmatrix} \to \begin{pmatrix} 1&1&1&1 \\ 0&-1&0&-1 \\ 0&1&0&1 \end{pmatrix} \to \begin{pmatrix} 1&1&1&1 \\ 0&-1&0&-1 \\ 0&0&0&0 \end{pmatrix}$ via $R_2 - R_1 \to R_2, R_2 + R_3 \to R_3$. Hence a basis for U_2 is $\{(1,1,1,1), (0,-1,0,-1)\}$.

   Alternate solution: We row reduce $\begin{pmatrix} 1&1&0 \\ 1&0&1 \\ 1&1&0 \\ 1&0&1 \end{pmatrix} \to \begin{pmatrix} 1&1&0 \\ 0&-1&1 \\ 0&0&0 \\ 0&-1&1 \end{pmatrix} \to \begin{pmatrix} 1&1&0 \\ 0&-1&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ via $R_2 - R_1 \to R_2, R_3 - R_1 \to R_3, R_4 - R_1 \to R_4, R_4 - R_2 \to R_4$. The pivots are in the first and second columns, hence the first and second of u, v, w form a basis for U_2: namely $\{u, v\}$.

4. Find a basis for $U_1 + U_2$.

   We row reduce $\begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 1&1&1&1 \\ 1&0&1&0 \\ 0&1&0&1 \end{pmatrix} \to \begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&-1&1&0 \\ 0&1&0&1 \end{pmatrix} \to \begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&2&0 \\ 0&0&-1&1 \end{pmatrix} \to \begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&0&-2 \\ 0&0&0&2 \end{pmatrix} \to \begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&0&-2 \\ 0&0&0&0 \end{pmatrix}$ via $R_3 - R_1 \to R_3, R_4 - R_1 \to R_4, R_4 + R_2 \to R_4, R_5 - R_2 \to R_5, R_4 - 2R_3 \to R_4, R_5 + R_3 \to R_5, R_5 + R_4 \to R_4$. Hence a basis for $U_1 + U_2$ is $\{(1,1,0,0), (0,1,1,0), (0,0,1,1), (0,0,0,-2)\}$.

   Alternate solution: row reduce the transpose of the starting matrix. The pivots end up in the first four columns, so a basis is $\{r, s, u, v\}$.

5. Find $dim(U_1), dim(U_2), dim(U_1 + U_2), dim(U_1 \cap U_2)$.

   Counting the size of each basis, we find $dim(U_1) = 2, dim(U_2) = 2, dim(U_1 + U_2) = 4$. Using $dim(U_1) + dim(U_2) = dim(U_1 + U_2) + dim(U_1 \cap U_2)$, we find $dim(U_1 \cap U_2) = 0$. This is an example of two planes whose intersection is a single point (namely, $\overline{0}$). This phenomenon is impossible in three dimensions, but possible in 4 or more.