Math 254 Spring 2014 Exam 4 Solutions

1. Carefully state the definition of "span". Give two (different) sets of vectors from $M_{2,2}$, neither of which may have the vector I in their span.

   The span of a set of vectors $\{v_1, \ldots, v_k\}$ is the set of all linear combinations $\{a_1v_1 + \cdots + a_kv_k : a_i \in \mathbb{R}\}$. Many examples are possible; to receive full credit you must give a set containing vectors (which are matrices here). Three possible examples are $\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$.

2. Carefully state five of the eight vector space axioms.

   These are listed on p.152 of the text. To receive full credit, you need to include complete statements, such as "For all vectors u,v, u+v=v+u."

3. Prove that for any vector space V, and for any vector $\overline{v} \in V$, that $0\overline{v} = \overline{0}$.

   Since $0 + 0 = 0$ as real numbers, we have $0\overline{v} = (0 + 0)\overline{v} = (0\overline{v}) + (0\overline{v})$, using distributivity. We then add $-(0\overline{v})$ to both sides, getting $0\overline{v} - (0\overline{v}) = ((0\overline{v}) + (0\overline{v})) - (0\overline{v})$. We now simplify using associativity, the $u + (-u) = \overline{0}$ axiom, and the $u + \overline{0} = u$ axiom, getting $\overline{0} = 0\overline{v}$, as desired.

4. Let $V \subseteq P_2(t)$ be defined by $V = \{p(t) : \forall t,\ p(t) \geq p'(t)\}$, where $p'(t)$ denotes the derivative. Determine whether or not V is a vector space.

   V contains $\overline{0}$, and is closed under vector addition. However it is not closed under scalar multiplication. To prove this we need one specific counterexample. Take $p(t) = 5$. Since $p'(t) = 0 < p(t)$ for all t, we have $p(t) \in V$. However $(-1)p(t) = -5$ is not in V, because $p'(t) = 0 > -5$. Hence V is not closed under scalar multiplication and is not a vector space.

   Other choices for counterexample are possible, such as $p(t) = t^2 + 2$. Note that $p(t) - p'(t) = t^2 - 2t + 2 = (t - 1)^2 + 1$, which is positive for all t; hence $p(t) \in V$.

5. Set $V = P_2(t)$. Give any two subspaces U_1, U_2 such that $U_1 \oplus U_2 = V$. Be sure to justify.

   Many choices are possible: $U_1 = Span(1, t), U_2 = Span(t^2)$, or $V_1 = Span(1, t^2), V_2 = Span(t)$. To receive full credit, you must give two vector spaces, each contained in $P_2(t)$, whose sum is V and whose intersection is $\{0\}$, and you must justify all of these statements.