Math 254 Spring 2014 Exam 11 Solutions

1. Carefully state the definition of $P(t)$. Give a set of three vectors from $P(t)$.

The vector space $P(t)$ consists of all polynomials in the variable t, with real coefficients. A sample set of three vectors is $\{1, t, t^3 - t\}$.

2. You seek the Jordan Canonical Form for a 7×7 matrix. First, you find the characteristic polynomial $\Delta(t) = (t-5)^7$. Next, you determine that the geometric multiplicity for the eigenvalue $\lambda = 5$ is 3. With this information, what are the possible minimal polynomials $m(t)$? For each possibility, determine what combinations of Jordan blocks are possible.

There will be three Jordan blocks, of total size 7. Hence there are four possibilities (up to rearranging the blocks): $J(5,5)J(5,1)J(5,1)$ or $J(5,4)J(5,2)J(5,1)$ or $J(5,3)J(5,3)J(5,1)$ or $J(5,3)J(5,2)J(5,2)$. The first has minimum polynomial $m(t) = (t-5)^5$, where the exponent 5 reflects the size of the largest Jordan block. The second has $m(t) = (t-5)^4$; the last two both have $m(t) = (t-5)^3$.
To find JCF, we calculate $m(t)$ and hope it's not $(t-5)^3$, which would be ambiguous.

The remaining three problems all concern matrix $M = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{pmatrix}$.

3. Find the characteristic polynomial $\Delta(t)$ for M, and calculate each eigenvalue with its algebraic multiplicity.

We calculate $det(tI - M) = \begin{vmatrix} t & -1 & 1 \\ -2 & t & -2 \\ -3 & 2 & t-4 \end{vmatrix} = t^3 - 4t^2 + 5t - 2 = (t-2)(t-1)^2$. Hence we have $\lambda = 2$ with algebraic multiplicity 1, and $\lambda = 1$ with algebraic multiplicity 2.

4. Find the geometric multiplicity of each eigenvalue, and a corresponding maximal independent set of eigenvectors.

For $\lambda = 2$, the geometric multiplicity is automatically 1, and we have $2I - A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 2 & -2 \\ -3 & 2 & -2 \end{pmatrix}$ which row reduces to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Hence a basis for its eigenspace is $\{(0,1,1)\}$.

For $\lambda = 1$, we calculate $1I - A = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -2 \\ -3 & 2 & -3 \end{pmatrix}$ which row reduces to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence a basis for its eigenspace is $\{(1,0,-1)\}$ and the geometric multiplicity is 1.

5. Find the Jordan Canonical Form for M.

We have calculated that the JCF must be $J(1,2)J(2,1)$ (or $J(2,1)J(1,2)$), which correspond to matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (or $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$).