Math 254 Spring 2014 Exam 10 Solutions

1. Carefully state the definition of $M_{m,n}$. Give a set of two vectors from $M_{3,2}$.

   The vector space $M_{m,n}$ consists of all matrices with m rows and n columns (with real entries). A set of two vectors from $M_{3,2}$ is $\left\{\begin{pmatrix}1&1\\2&2\\3&3\end{pmatrix}, \begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix}\right\}$.

2. For $n \geq 1$, the "checkerboard" matrix A has entries $a_{i,j} = (-1)^{i+j}$. Find $|A|$.

   For an example, we consider $n = 3$. Here we have $A = \begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{pmatrix}$; we see that adding the second row to the first gives an all-zero row, so $|A| = 0$. In fact this is always true, since $(-1)^{1+j} + (-1)^{2+j} = (-1)^{1+j} - (-1)^{1+j} = 0$ for all j. Hence $|A| = 0$ for all checkerboard matrices, provided that $n \geq 2$. The case $n = 1$ is special: $A = (1)$, so $|A| = 1$.

   Alternate solution: For $n \geq 3$, the first and third rows of A are the same, so $|A| = 0$. We consider the special cases $n = 1, 2$. For $n = 1$, $|A| = 1$; for $n = 2$, $|A| = 0$.

The remaining three problems all concern matrix $M = \begin{pmatrix}0&1&-2\\1&2&3\\2&-3&5\end{pmatrix}$.

3. Find $|M|$ using the formula for 3×3 determinants.

   We use the mnemonic that begins by repeating the first two columns of M. This gives $\begin{pmatrix}0&1&-2&0&1\\1&2&3&1&2\\2&-3&5&2&-3\end{pmatrix}$. We now calculate $|M| = 0 \cdot 2 \cdot 5 + 1 \cdot 2 \cdot 3 + (-2) \cdot 1 \cdot (-3) - (2 \cdot 2 \cdot (-2)) - ((-3) \cdot 3 \cdot 0) - (5 \cdot 1 \cdot 1) = 0 + 6 + 6 + 8 + 0 - 5 = 15$.

4. Find $|M|$ with a Laplace expansion on the first column.

   We have $|M| = 0\begin{vmatrix}2&3\\-3&5\end{vmatrix} - 1\begin{vmatrix}1&-2\\-3&5\end{vmatrix} + 2\begin{vmatrix}1&-2\\2&3\end{vmatrix} = 0 - 1(1\cdot 5 - (-3)\cdot(-2)) + 2(1\cdot 3 - 2\cdot(-2)) = 1 + 14 = 15$.

5. Find $|M|$ using elementary row operations to make it block triangular.

   Method 1: Swap rows 1,3 to get $\begin{pmatrix}2&-3&5\\1&2&3\\0&1&-2\end{pmatrix}$. This multiplies the determinant by -1. Then $R_2 - \frac{1}{2}R_1 \to R_2$ gives $\begin{pmatrix}2&-3&5\\0&3.5&0.5\\0&1&-2\end{pmatrix}$. This does not change the determinant. We now have a block triangular matrix, whose determinant is $|2|\begin{vmatrix}3.5&0.5\\1&-2\end{vmatrix} = 2\cdot(-7-0.5) = -15$. Hence $|A| = -(-15) = 15$.

   Method 2: We continue one more step, via $R_3 = R_3 - \frac{2}{7}R_2$ to get $\begin{pmatrix}2&-3&5\\0&3.5&0.5\\0&0&-\frac{15}{7}\end{pmatrix}$. This is now fully triangular (and also block triangular). We now find its determinant by multiplying along the diagonal, i.e. $2 \cdot \frac{7}{2} \cdot \frac{-15}{7} = -15$. Hence $|A| = -(-15) = 15$.