Math 254 Spring 2014 Exam 0 Solutions

1. Carefully state the definition of vector space $M_{2,2}$. Give two example vectors.

   $M_{2,2}$ is the set of all 2×2 matrices with real entries. Some examples: $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$.

2. Carefully state the definition of "spanning". Give two examples from $P_1(t)$.

   A set of vectors S, drawn from vector space V, is spanning if $Span(S) = V$. Examples from $P_1(t)$ include $\{1, t\}$, $\{1, t, 2t\}$, $\{1, 1+t\}$.

3. Consider the subset of $\mathbb{R}^2$ given by $S = \{(1, 0), (2, 3)\}$. Prove that S is spanning.

   Let $(a, b) \in \mathbb{R}^2$ be arbitrary. After a side calculation, not shown, we use the linear combination $\frac{3a-2b}{3}(1, 0) + \frac{b}{3}(2, 3) = (a - \frac{2b}{3}, 0) + (\frac{2b}{3}, b) = (a, b)$. Hence $\mathbb{R}^2 \subseteq Span(S)$, so S is spanning.

   A correct solution must prove that *all* elements of $\mathbb{R}^2$ may be achieved as linear combinations from S.

4. Consider the subset of $\mathbb{R}^2$ given by $T = \{(x, y) : \sin x = \sin y\}$. Prove that T is not closed.

   Note that $(0, \pi) \in T$ since $\sin 0 = 0 = \sin \pi$. However, $\frac{1}{2}(0, \pi) = (0, \frac{\pi}{2}) \notin T$ since $\sin 0 = 0 \neq 1 = \sin \frac{\pi}{2}$. Hence T is not closed under scalar multiplication.

   Another approach is to note that $(0, \pi) \in T$ as before, and that $(\frac{\pi}{2}, \frac{\pi}{2}) \in T$. However their sum is $(0, \pi) + (\frac{\pi}{2}, \frac{\pi}{2}) = (\frac{\pi}{2}, \frac{3\pi}{2}) \notin T$ since $\sin \frac{\pi}{2} = 1 \neq -1 = \sin \frac{3\pi}{2}$. Hence T is not closed under vector addition.

   A correct solution must include a specific counterexample, with numbers, demonstrating that closure fails.

5. Consider the polynomial space $P(t)$. Prove that $\{t^2 + 3, t^2 + 4, t^2 + 5\}$ is dependent.

   After a side calculation, not shown, we use the nondegenerate linear combination $1(t^2 + 3) - 2(t^2 + 4) + 1(t^2 + 5) = 0$. Hence the set is dependent.

   A correct solution must include a specific nondegenerate linear combination, with numbers, yielding 0.