Math 254 Fall 2014 Exam 9 Solutions

1. Carefully state the definition of "spanning". Give two spanning sets for $\mathbb{R}^2$.

   A set of vectors is spanning if its span is the entire vector space. Examples are $\{(1,0),(0,1)\}$ and $\{(1,1),(1,-1)\}$.

2. Suppose that U is a vector space with (finite) basis B. Suppose that F, G are two linear transformations from U to U. Prove that if $[F]_B = [G]_B$ then $F = G$.

   Solution 1: To prove two functions are equal, they have to have the same domain, and agree on each element of its domain. Both F, G have domain U. Let $u \in U$. We have $[F(u)]_B = [F]_B[u]_B = [G]_B[u]_B = [G(u)]_B$. Hence $F(u), G(u)$ have the same representations, and must be equal.

   Solution 2: $[F]_B, [G]_B$ are similar, since $[F]_B = I[G]_B I^{-1}$. Hence they represent the same linear transformation, so $F = G$.

The remaining three problems concern the vector space $V = \{\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) : a, b, d \in \mathbb{R}\}$, its basis $E = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$, and $F : V \to V$ given by $F : \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) \to \left(\begin{smallmatrix} d & a+d-b \\ a+d-b & a \end{smallmatrix}\right)$.

3. Prove that $F^2 = F \circ F$ is the identity linear transformation.

   Solution 1: $F(F(\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right))) = F(\left(\begin{smallmatrix} d & a+d-b \\ a+d-b & a \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & d+a-(a+d-b) \\ d+a-(a+d-b) & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$.

   Solution 2: From (4), we have $[F]_E = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. We calculate $[F]_E[F]_E = I_3$; hence $[F \circ F]_E = I_3$ and so F^2 must be the identity map.

4. Calculate $[F]_E$.

   We calculate: $F(e_1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ so $[F(e_1)]_E = (0,1,1)$, $F(e_2) = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ so $[F(e_2)]_E = (0,-1,0)$, $F(e_3) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ so $[F(e_3)]_E = (1,1,0)$. Combining, we get $[F]_E = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$.

5. Find the row canonical form of $[F]_E$, and use this to determine the rank and nullity of F.

   Starting with $[F]_E = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, we subtract the first and last rows from the middle row, then swap the first and last rows, and multiply the middle row by -1. The result is I_3, which has three pivots and no columns without pivots. Hence the rank of F is 3, and the nullity is zero.