Math 254 Fall 2014 Exam 7 Solutions

1. Carefully state the definition of "linearly dependent". Give two different linearly dependent sets from $P_1(t)$.

   A set of vectors is linearly dependent if a nondegenerate linear combination yields the zero vector. Examples: $\{1,2\},\{1,t,1+t\}$. Common errors: not a set, not form $P_1(t)$.

2. Prove that $f(x)=x^{-1/2}$ is in $L^1(0,1)$ but not in $L^3(0,1)$.

   L^1: $\int_0^1 |f(x)|^1 dx = \int_0^1 x^{-1/2}dx = \lim_{m\to 0+} 2x^{1/2}|_m^1 = \lim_{m\to 0+} 2-2\sqrt{m} = 2$. Since this is finite, $f(x)\in L^1(0,1)$.
   L^3: $\int_0^1 |f(x)|^3 dx = \int_0^1 x^{-3/2}dx = \lim_{m\to 0+} -2x^{-1/2}|_m^1 = \lim_{m\to 0+} -2+\frac{2}{\sqrt{m}} = \infty$. Since this doesn't converge, $f(x)\notin L^3(0,1)$.

The remaining problems all concern the inner product on $\mathbb{R}^3$ defined by $\langle x,y\rangle_A = x^T A y$, where A is the positive definite matrix $\begin{pmatrix} 3&0&0\\0&2&1\\0&1&2\end{pmatrix}$. Set $u=(1,1,-2)^T, v=(1,2,-1)^T$.

3. Calculate $\langle u,u\rangle_A, \langle v,v\rangle_A, \langle u,v\rangle_A$.

   $\langle u,u\rangle_A = \begin{pmatrix}1&1&-2\end{pmatrix}\begin{pmatrix} 3&0&0\\0&2&1\\0&1&2\end{pmatrix}\begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}1&1&-2\end{pmatrix}\begin{pmatrix}3\\0\\-3\end{pmatrix} = 9.$
   $\langle v,v\rangle_A = \begin{pmatrix}1&2&-1\end{pmatrix}\begin{pmatrix} 3&0&0\\0&2&1\\0&1&2\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1&2&-1\end{pmatrix}\begin{pmatrix}3\\3\\0\end{pmatrix} = 9.$
   $\langle u,v\rangle_A = \begin{pmatrix}1&1&-2\end{pmatrix}\begin{pmatrix} 3&0&0\\0&2&1\\0&1&2\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1&1&-2\end{pmatrix}\begin{pmatrix}3\\3\\0\end{pmatrix} = 6.$

4. Use the Gram-Schmidt process to find an orthogonal basis for $W=Span(u,v)$. Note: orthogonal means in the $\langle\cdot,\cdot\rangle_A$ sense.

   We build orthogonal basis $\{w_1,w_2\}$ as follows: set $w_1=u$, and $w_2 = v-proj(v,w_1) = v-\frac{\langle v,w_1\rangle_A}{\langle w_1,w_1\rangle_A}w_1 = v-\frac{\langle v,u\rangle_A}{\langle u,u\rangle_A}u$. We calculated these in (3), so $w_2=(1,2,-1)^T - \frac{6}{9}(1,1,-2)^T = (\frac{1}{3},\frac{4}{3},\frac{1}{3})^T$. If desired, we can double-check that $\langle w_1,w_2\rangle_A=0$.

5. Let $W=Span(u,v)$. Find a nonzero vector in $W^\perp$. Note: This means in the $\langle\cdot,\cdot\rangle_A$ sense.

   Solution 1: Pick any vector, say $z=(1,0,0)$, not in W. Using the orthogonal basis from (4), we calculate $z-proj(z,w_1)-proj(z,w_2)$. We get $z-\frac{3}{9}(1,1,-2)-\frac{1}{5}(\frac{1}{3},\frac{4}{3},\frac{1}{3}) = (\frac{3}{5},-\frac{3}{5},\frac{3}{5})$. If desired we can rescale to $(1,-1,1)$.

   Solution 2: We seek $z=(a,b,c)$ to satisfy $\langle z,u\rangle_A = \langle z,v\rangle_A = 0$. This gives the homogeneous linear system $\{3a-3c=0, 3a+3b=0\}$, which has two pivots and a one-dimensional solution space. We can choose any one element from this space, such as $(1,-1,1)$.