Math 254 Fall 2014 Exam 6 Solutions

1. Carefully state the definition of "polynomial space" $P(t)$. Give two different bases for $P_1(t)$.

   The polynomial space $P(t)$ consists of all polynomials, with real coefficients, in the variable t. Two bases for $P_1(t)$ are $\{1, t\}$ and $\{1+t, 1-t\}$.

2. Let V denote the set of all symmetric 2×2 matrices. Set $E = \{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$. Prove that E is a basis for V.

   We first prove that E is independent: If $ae_1 + be_2 + ce_3 = 0$, then $\left(\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, so $a = b = c = 0$. Hence no nondegenerate linear combination yields 0.
   Solution 1: Let $\left(\begin{smallmatrix}a&b\\b&d\end{smallmatrix}\right) \in V$. We take $ae_1 + be_2 + ce_3$, and see that it equals the desired matrix. Hence E is spanning, and hence E is a basis.
   Solution 2: $V \neq M_{2,2}$ so $dim(V) \leq 3$. But E is independent and $|E| = 3$, so E is maximal spanning, and is thus a basis.

The remaining three problems concern the vector space $V = \{\left(\begin{smallmatrix}a&b\\b&d\end{smallmatrix}\right) : a, b, d \in \mathbb{R}\}$ and its basis $E = \{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$.

3. Set $B = \{\left(\begin{smallmatrix}0&-2\\-2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\} = \{b_1, b_2, b_3\}$. Compute $[b_1]_E, [b_2]_E, [b_3]_E$, and use these to prove that B is a basis for V.

   We have $[b_1]_E = (0, -2, 1), [b_2]_E = (0, 1, 0), [b_3]_E = (1, 0, -1)$. Putting these as rows, we get $\left(\begin{smallmatrix}0&-2&1\\0&1&0\\1&0&-1\end{smallmatrix}\right)$. After $R_1 + 2R_2 \to R_1, R_1 \leftrightarrow R_3$, we get $\left(\begin{smallmatrix}1&0&-1\\0&1&0\\0&0&1\end{smallmatrix}\right)$. This is in row echelon form, and has 3 pivots, so the rowspace of the original matrix is 3-dimensional. Hence $dim(Span(B)) = 3 = dim(V)$, and thus $Span(B) = V$, and B is a basis for V.

4. Set $C = \{\left(\begin{smallmatrix}1&3\\3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}2&6\\6&4\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}5&3\\3&4\end{smallmatrix}\right)\} = \{c_1, c_2, c_3, c_4\}$. Compute $[c_1]_E, [c_2]_E, [c_3]_E, [c_4]_E$, and use these to find a basis for $Span(C)$.

   We have $[c_1]_E = (1, 3, 2), [c_2]_E = (2, 6, 4), [c_3]_E = (1, 1, 1), [c_4]_E = (5, 3, 4)$. Putting these as rows, we get $\left(\begin{smallmatrix}1&3&2\\2&6&4\\1&1&1\\5&3&4\end{smallmatrix}\right)$, which has row echelon form $\left(\begin{smallmatrix}1&3&2\\0&2&1\\0&0&0\\0&0&0\end{smallmatrix}\right)$. Hence $Span(C)$ has basis $\{\left(\begin{smallmatrix}1&3\\3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}0&2\\2&1\end{smallmatrix}\right)\}$.

5. For B as in (3), calculate Q_{BE}, and use this to compute $[\left(\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right)]_B$.

   We now put the $[b_1]_E, [b_2]_E, [b_3]_E$ as columns, to get Q_{EB}. We calculate $Q_{BE} = Q_{EB}^{-1} = \left(\begin{smallmatrix}1&0&1\\2&1&2\\1&0&0\end{smallmatrix}\right)$. Since $[\left(\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right)]_E = (1, 2, 3)^T$, we calculate $[\left(\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right)]_B = Q_{BE}(1, 2, 3)^T = (4, 10, 1)^T$. That is, $\left(\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right) = 4b_1 + 10b_2 + 1b_3$, which is easily double-checked if desired.