Math 254 Fall 2013 Exam 0 Solutions

1. Carefully state the definition of "matrix space". Give two example vectors from $M_{2,3}$.
   A matrix space $M_{m,n}$, for positive integers m, n, is the set of all matrices with m rows and n columns.

2. Let $u = [\,1\ 2\ 7\,]$, $v = u^T$. For each of the following, determine what type they are (undefined, scalar, matrix/vector). For each matrix/vector, specify the dimensions.

   (a) $u + (u \cdot v)$ Undefined

   (b) $u + v^T$ Row 3-vector, or 1×3 matrix

   (c) vu 3×3 matrix

   (d) uv 1×1 matrix, or scalar

   (e) $(u \times v) \times v$ 3-vector

3. Let $u = (1, 1, 1, 3), v = (-1, 0, 1, 1)$. Find the angle between u, v.
   The desired angle, θ, satisfies $\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{u\cdot v}{\sqrt{u\cdot u}\sqrt{v \cdot v}} = \frac{3}{\sqrt{12}\sqrt{3}} = \frac{3}{\sqrt{36}} = \frac{1}{2}$. Fortunately, this is a familiar angle, $\theta = \pi/3$. Children sometimes call this $60°$.

4. Determine, with justification, all possible x (if any) that makes the following hold:
   $$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & x \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
   We calculate $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & x \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+2 \\ 0 & x+3 \end{bmatrix}$. Hence two of the four entries are always what we want, and the others are too, provided $x + 2 = 0$ and $x + 3 = 1$. This happens for $x = -2$, and only then.

5. For $u = (1, 0, 1), v = (-1, 1, 0)$, calculate $u \times v$.
   Solution 1: We write $\begin{vmatrix} i & j & k & i & j \\ 1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 1 \end{vmatrix}$ to calculate the determinant, as $-j + k - i = (-1, -1, 1)$.
   Solution 2: $u \times v = (i+k) \times (-i+j) = -(i \times i) + (i \times j) - (k \times i) + (k \times j) = k - j - i = (-1, -1, 1)$.

Extra: We say that the *inverse* of a 2×2 square matrix, is another 2×2 square matrix, with their product the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. You just determined an inverse for $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.
(1) Prove that if the rows of a matrix A are the same, then A does **NOT** have an inverse.
(2) Find all a, b, c such that $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is its own inverse.

(1) Since the rows of A are the same, the rows of AB are the same, for any matrix B (because of the way we multiply matrices). However, the rows of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are not the same, so AB can't equal the identity.
(2) We have $MM = \begin{bmatrix} a^2 & b(a+c) \\ 0 & b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $a^2 = b^2 = 1$. Also $b(a + c) = 0$ so either $b = 0$ or $a + c = 0$. This gives four solutions $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$, and also the two infinite families of solutions $\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$ (for any b), and $\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$ (for any b).