Math 254 Exam 9 Solutions

1. Carefully state the definition of "spanning". Give two examples, each from $P_2(t)$.

   A set of vectors is spanning if every vector in the vector space may be achieved as a linear combination of vectors from this set. Many examples are possible, such as $\{1, t, t^2\}, \{1, 2t, 3t^2\}, \{1, t, t^2, t + t^2\}, \{1, t + 1, t^2 + 1\}$.

2. Consider the basis $S = \{(-1, -2), (2, 5)\}$ of $\mathbb{R}^2$, and the linear operator $F(x, y) = (-2x + 2y, -10x + 7y)$. Find the matrix representation $[F]_S$.

   We have $P_{ES} = \begin{pmatrix} -1 & 2 \\ -2 & 5 \end{pmatrix}$, so $P_{SE} = P_{ES}^{-1} = \begin{pmatrix} -5 & 2 \\ -2 & 1 \end{pmatrix}$. We calculate $[F]_E = ([F(e_1)]_E[F(e_2)]_E) = ([(-2, -10)]_E[(2, 7)]_E) = \begin{pmatrix} -2 & 2 \\ -10 & 7 \end{pmatrix}$. Hence $[F]_S = P_{SE}[F]_E P_{ES} = \begin{pmatrix} -5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ -10 & 7 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

3. Prove that, for all square matrices A, that A must be similar to A.

   Set $P = I$, the identity matrix. We have $P^{-1} = P = I$. Now, $A = IAI = P^{-1}AP$, so A is similar to A.

4. Let V be the vector space of functions that have as a basis $S = \{e^t, te^t, t^2e^t\}$. Find the matrix representation $[\frac{d}{dt}]_S$.

   We first calculate, using the product rule, $\frac{d}{dt}(e^t) = e^t$, $\frac{d}{dt}(te^t) = te^t+e^t$, $\frac{d}{dt}(t^2e^t) = t^2e^t + 2te^t$. Hence $[\frac{d}{dt}]_S = ([e^t]_S[te^t + e^t]_S[t^2e^t + 2te^t]_S) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

5. Set $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Prove that A is not similar to B.

   Solution 1: We use the theorem that if A is similar to B then the determinant and trace of A, B must be the same. If either one disagrees on A, B, then A is not similar to B. As it happens, both disagree, so either choice will work. Determinant: $det(A) = 1\cdot2-1\cdot1 = 1$, while $det(B) = 1\cdot1-1\cdot(-1) = 2$, so A is not similar to B. Trace: $trace(A) = 1+2 = 3$, while $trace(B) = 1+1 = 2$, so A is not similar to B.

   Solution 2: Suppose A is similar to B. Then there exists some P with $A = P^{-1}BP$, and we multiply by P to get $PA = BP$ (this step isn't necessary, but it saves us finding P^{-1}). Set $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $PA = \begin{bmatrix} a+b & a+2b \\ c+d & c+2d \end{bmatrix}$ and $BP = \begin{bmatrix} a-c & b-d \\ a+c & b+d \end{bmatrix}$. Setting these equal, we get $a + b = a - c, a + 2b = b - d, c + d = a + c, c + 2d = b + d$. The first equation gives $b = -c$, while the third gives $a = d$. Plugging into the second equation gives $c = 2a$. Plugging all this into the last equation gives $a = 0$. But $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ isn't invertible, so A couldn't have been similar to B.