Math 254 Exam 5 Solutions

1. Carefully state the definition of "independent". Give two examples, each from $\mathbb{R}^3$.

   A set of vectors is independent if there is no nondegenerate linear combination yielding the zero vector. Many examples are possible, such as $\{(1,2,3)\}$ or $\{(1,2,3),(1,1,1)\}$.

2. Give any two bases for $M_{2,2}(\mathbb{R})$, the set of 2×2 matrices.

   Unfortunately, many students were challenged by this question. A basis is a set of vectors (with certain properties), which in this context is a set of 2×2 matrices. Anything that is NOT a set of matrices cannot possibly be a basis of this vector space.

   Since this space is 4-dimensional, each basis must contain four matrices, that are linearly independent and spanning. Many examples are possible, such as $\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$ (the standard basis), or $\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$, or $\{\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)\}$.

The following problems are all in $\mathbb{R}^4$, and concern the subspaces $S = \{(a, b-4a, c-2a, a) : a,b,c \in \mathbb{R}\}$ and $T = Span(\{(1,-4,-2,1),(1,-3,-1,2),(3,-8,-2,7)\})$.

3. Find a basis for S, and a basis for T.

   S is at most three dimensional since there are three free variables, so we try to find three linearly independent elements of S. We try $a=1, b=c=0$ for $(1,-4,-2,1)$; $b=1, a=c=0$ for $(0,1,0,0)$; $c=1, a=b=0$ for $(0,0,1,0)$. These three vectors, placed as rows into a matrix, are already in echelon form, so are independent and hence a basis of S. For T, we calculate row echelon form for $\begin{pmatrix}1&-4&-2&1\\1&-3&-1&2\\3&-8&-2&7\end{pmatrix} \to \begin{pmatrix}1&-4&-2&1\\0&1&1&1\\0&4&4&4\end{pmatrix} \to \begin{pmatrix}1&-4&-2&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$ so a basis is $\{(1,-4,-2,1),(0,1,1,1)\}$.

   Note: Unfortunately, some students had trouble with S. S is a set of infinitely many vectors (for each possible value of a,b,c). It is equivalent to $\{a(1,-4,-2,1)+b(0,1,0,0)+c(0,0,1,0) : a,b,c \in \mathbb{R}\} = Span(\{(1,-4,-2,1),(0,1,0,0),(0,0,1,0)\})$. Writing S in this way is not necessary, but perhaps may be helpful for anyone still confused.

4. Find a basis for $S+T$.

   We put the five basis vectors (or the basis for S together with the three spanning T) together, then find the row echelon form. $\begin{pmatrix}1&-4&-2&1\\0&1&0&0\\0&0&1&0\\1&-4&-2&1\\0&1&1&1\end{pmatrix} \to \begin{pmatrix}1&-4&-2&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{pmatrix} \to \begin{pmatrix}1&-4&-2&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$. Hence, one possible basis for $S+T$ is given by $\{(1,-4,-2,1),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$. Of course, this is in $\mathbb{R}^4$, so $S+T=\mathbb{R}^4$ since they are each 4-dimensional, so any basis of $\mathbb{R}^4$ will do, such as the standard basis.

5. Find $dim(S \cap T)$ and find a basis for $S \cap T$.

   We have $dim(S+T) + dim(S \cap T) = dim(S) + dim(T)$. From the earlier problems, we have $4 + dim(S \cap T) = 3 + 2$, so $dim(S \cap T) = 1$. By inspection, we see that $(1,-4,-2,1)$ is in both S and T, so $\{(1,-4,-2,1)\}$ is a basis for $S \cap T$.