Math 254 Exam 4 Solutions

1. Carefully state the definition of "subspace". Give two examples, each from $\mathbb{R}^2$.

   A subspace is a subset of a vector space that is, itself, a vector space. Many examples are possible, such as $\{(0,0)\}$ (zero-dimensional), Span(S) for $S = \{(1,2)\}$ (one-dimensional), or $\mathbb{R}^2$ itself (two-dimensional).

2. Carefully state any five of the eight vector space axioms.

   These are listed on p.152 of the text. It is not important how you number them; however it is important that you give the English text correctly. "For any vectors u, v, w in V, $(u+v)+w = u+(v+w)$." is correct, but the equation "$(u+v)+w = u+(v+w)$" alone is incorrect.

3. Let $S = \{f(x) : f(3) = 1\} \subseteq \mathbb{R}[x]$ be the set of all polynomials $f(x)$ satisfying $f(3) = 1$. Determine, with justification, whether this is a vector space.

   Since S is a subset of a vector space, to be a subspace S must satisfy three properties. It must contain the zero vector, it must be closed under vector addition, and closed under scalar multiplication. S satisfies *none* of these three properties, and it's enough to pick your favorite to disprove. Just for fun, I will disprove all three: (1) $f(x) = 0$ does not satisfy $f(3) = 1$, so 0 is not in S; (2) $f(x) = 1$ and $g(x) = x/3$ are both in S, but $(f+g)(x) = 1+x/3$ is not in S since $(f+g)(3) = 2$; (3) $f(x) = 1$ is in S but $5f(x) = 5$ is not in S.

4. Determine, with justification, whether $(1,2)$ is in the rowspace of $M = \left[\begin{smallmatrix} 2 & 3 \\ 6 & 9 \end{smallmatrix}\right]$.

   The rowspace of M is also the rowspace of $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right]$, obtained via $R_2 = R_2 - 3R_1$, which is $Span((2,3)) = \{t(2,3) : t \in \mathbb{R}\}$. If $(1,2)$ were in this subspace, then for some t we would have $(1,2) = (2t, 3t)$, and hence $2t = 1$ and $3t = 2$. This is impossible, so the answer is "no".

5. Set $V = \mathbb{R}^3$. Give any two subspaces U_1, U_2 such that $U_1 \oplus U_2 = V$.

   Two type of solutions are possible. The "trivial" solution is $U_1 = \{(0,0,0)\}$, $U_2 = \mathbb{R}^3$ (or the other way around). Otherwise, one of U_1, U_2 will be one-dimensional and the other will be two-dimensional. Many examples are possible, for example $U_1 = Span(\{(1,0,0)\}) = \{(a,0,0) : a \in \mathbb{R}\}, U_2 = Span(\{(0,1,0),(0,0,1)\}) = \{(0,b,c) : b,c \in \mathbb{R}\}$. What is important is that U_1, U_2 are both subspaces of R^3, that $U_1 + U_2 = \mathbb{R}^3$, and that $U_1 \cap U_2 = \{0\}$.