Math 254 Final Exam: 12/12/6

Please read the exam instructions.

Notes, books, papers, calculators and electronic aids are all forbidden for this exam, with the exception of a single formula page. Please write your answers on **separate paper**, indicate clearly what work goes with which problem, and put your name on every sheet. Cross out work you do not wish graded; incorrect work can lower your grade, even compared with no work at all. Keep this list of problems for your records. Show all necessary work in your solutions; if you are unsure, show it. Each problem is worth 10 points. There are 15 questions.

1. Carefully define the term "dimension" as it applies to vector spaces. Give two examples: a four-dimensional vector space, and an infinite-dimensional vector space.

2. Carefully define the term "basis". Give two examples in $\mathbb{R}^2$.

The following two questions involve the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by $f(x, y, z) = (x - y, y - z, z - x, x + z - 2y)$.

3. Represent f as a matrix multiplication.

4. Determine the rank and nullity of f.

The following two questions involve the matrix $A = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

5. Use Gaussian elimination to put A into echelon form.

6. Find all solutions to the linear system $A[p\ q\ r\ s]^T = [1\ 1\ 1]^T$.

7. Find all solutions to the following system of linear equations.

$$\begin{aligned} 4u - 3w &= 0 \\ -2u + 3v + 2w &= \text{-1} \\ 6u - 6v - 6w &= 1 \end{aligned}$$

8. Give three examples of 2×2 systems of linear equations; one each with zero, one, and infinitely many solutions. Justify your answers, providing all solutions to each system.

NOTE: THIS EXAM HAS 15 QUESTIONS; TURN THE PAGE.

9. Set $U = \{(a, b, c) : a + b = 2c;\ a, b, c \text{ are real}\}$. U is a subset of $\mathbb{R}^3$. Give three vectors from U, and determine whether or not U is a vector space.

10. Set $V = \mathbb{R}^5$. Give any two subspaces U_1, U_2 such that $U_1 \oplus U_2 = V$.

11. For the vector space $\mathbb{R}^3$, set $R = \text{span}\{(1, 2, 3)\}$. Find an orthogonal basis for $R^{\perp}$.

12. In the vector space $\mathbb{R}^3$, set $S = \text{span}\{(1, 2, 3), (5, -2, 3)\}$, $T = \text{span}\{(-1, 2, 1)\}$. Determine the dimensions of each of S, T, $S \cap T$, $S + T$. Justify your answers.

13. Let W be the vector space of functions that have as a basis $S = \{1, \sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta\}$. Let D be the differential operator on W. Find the matrix representation $[D]_S$.

The following two questions involve the matrix $B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

14. Find the characteristic polynomial $\Delta_B(\lambda)$ (or, if you prefer, $\Delta_B(t)$); use this to determine B's eigenvalues.

15. Find some P such that $P^{-1}BP$ is diagonal. NOTE: It is sufficient to give P; you need not calculate P^{-1}.