Math 254 Exam 10 Solutions

1. Carefully define the term "linear mapping (transformation)". Give two examples in $\mathbb{R}^2$.

   A function f whose domain and codomain are vector spaces. In addition, it must satisfy two properties: for any vectors u, v and scalar k, $f(u+v) = f(u)+f(v)$, $f(ku) = kf(u)$. Three examples in $\mathbb{R}^2$ are: $f(x,y) = (x,y)$, $f(x,y) = (3x+y, -y)$, $f(x,y) = (0,x)$.

   For the next three problems, consider the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \\ -1 & -2 & -3 \end{bmatrix}$.

2. Calculate $|A|$ by using the formula for 3×3 determinants.

   $|A| = -6 + 1 + (-6) - (-3) - 4 - (-3) = -9.$

3. Calculate $|A|$ by expanding on the second column.

   $|A| = (-1)1 \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} + (+1)1 \begin{vmatrix} 2 & 3 \\ -1 & -3 \end{vmatrix} + (-1)(-2) \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -(-3-1) + (-6+3) + 2(-2-3) = -9$

4. Calculate $|A|$ by making A triangular with elementary operations.

   $-2R_2 + R_1 \to R_1, R_2 + R_3 \to R_3, R_1 \leftrightarrow R_2, -R_2 + R_3 \to R_3$: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & -9 \end{bmatrix}$.

   This matrix has determinant $(1)(-1)(-9) = 9$. None of the operations we did change the determinant, apart from swapping two rows, which multiplied it by (-1). Hence the determinant is $9/(-1) = -9$.

5. Calculate $|B|$, for $B = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 1 & 0 & 2 & 0 & -2 \\ 0 & -1 & 2 & 5 & 0 \\ -2 & 0 & 0 & 2 & 3 \\ 3 & 0 & 1 & 0 & 1 \end{bmatrix}$.

   Many paths to the solution are possible; the simplest follows. $2R_3 + R_1 \to R_1$ yields a matrix with only one nonzero entry in the second column; we expand on that second column to get $|B| = (-1)(-1)|C|$. In matrix C, we perform $-7R_3 + R_1 \to R_1$ to get a matrix with only one nonzero entry in the third column; we expand on that third column to get $|C| = (+1)(2)|D|$. We find $|D| = 116$ using the 3×3 determinant formula. Hence $|C| = 2|D| = 232$, and $|B| = |C| = 232$.

   $C = \begin{bmatrix} 1 & 7 & 14 & 0 \\ 1 & 2 & 0 & -2 \\ -2 & 0 & 2 & 3 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 15 & 7 & -21 \\ 1 & 2 & -2 \\ 3 & 1 & -1 \end{bmatrix}$