MATH 245 S17, Exam 3 Solutions

1. Carefully define the following terms: recurrence, order of a recurrence, big Theta, set equality.

   A recurrence is a sequence in which all but finitely many terms are defined in terms of its previous terms. The order of a recurrence is the number of steps back in the recurrence that need to be known to compute each term. Given two sequences a_n, b_n, we say that a_n is big Theta of b_n if both $a_n = O(b_n)$ and $b_n = O(a_n)$. Two sets are equal if they contain the same elements.

2. Carefully define the following terms: Associativity of $\cup$ Theorem, De Morgan's Law for Sets Theorem, power set, Cantor's Theorem.

   The Associativity of $\cup$ Theorem says that for any sets R, S, T, we have $R \cup (S \cup T) = (R \cup S) \cup T$. The De Morgan's Law for Sets Theorem says that for any sets R, S, U with $R \subseteq U$ and $S \subseteq U$, both $(R \cup S)^c = R^c \cap S^c$ and $(R \cap S)^c = R^c \cup S^c$. Given a set S, the power set of S is the set whose elements are all the subsets of S. Cantor's Theorem says that for any set S, S is not equicardinal with its power set 2^S.

3. Let S, T be sets. Prove that $S \setminus T \subseteq S$.

   Let $x \in S \setminus T$. Hence $x \in S \wedge x \notin T$. By simplification, $x \in S$.

4. Prove that $n + 100 = O(n)$. Note that the Classification Theorem does not help.

   We need specific choices of n_0, M; many solutions are possible. One choice is $n_0 = 50, M = 3$. Now, let $n \geq n_0 = 50$. We have $|n + 100| = n + 100 \leq n + 2n = 3n = 3|n|$.

5. Suppose an algorithm has runtime specified by recurrence relation $T_n = 5T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

   In the notation of the Master Theorem, $a = 5, b = 2, c_n = n^2$. We calculate $d = \log_2 5$, and note that $d > \log_2 4 = 2$. Hence, we can take $d' = 2 < d$. Certainly $c_n = n^2 = O(n^2) = O(n^{d'})$. Hence the "small c_n" case of the Master Theorem applies, telling us that $T_n = \Theta(n^d) = \Theta(n^{\log_2 5})$.

6. Let S, T be sets. Prove that $S \times T$ is equicardinal with $T \times S$.

   We need to find an explicit pairing of $S \times T$ with $T \times S$. The natural one is $(x, y) \leftrightarrow (y, x)$, for every $x \in S$ and $y \in T$. In Chapter 13 we will have the tools to *prove* that this is a pairing; for now finding it is enough.

7. Set $R = \{1, 2, 3, 4, 5\}, S = \{4, 5, 6, 7\}, U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Calculate $|(R^c \cup S)^c \cup (S^c \setminus R)^c|$. Be sure to justify your answer.

   For convenience, let $[a, b]$ denote all the integers between a and b, inclusive. Step by step: $R^c = [6, 10]$. $R^c \cup S = [4, 10]$. $(R^c \cup S)^c = [1, 3]$. Now, $S^c = [1, 3] \cup [8, 10]$. $S^c \setminus R = [8, 10]$. $(S^c \setminus R)^c = [1, 7]$. Finally $(R^c \cup S)^c \cup (S^c \setminus R)^c = [1, 7]$, so the answer is $|[1, 7]| = |\{1, 2, 3, 4, 5, 6, 7\}| = 7$.

8. Solve the recurrence defined as $a_0 = a_1 = 2$, $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 2)$.

The characteristic equation is $r^2 = 4r - 4$, which factors as $(r-2)^2 = 0$. Hence there is a double root, and the general solution is $a_n = A2^n + Bn2^n$. We use the initial conditions to get $2 = a_0 = A2^0 + B \cdot 0 \cdot 2^0 = A$, and $2 = a_1 = A2^1 + B \cdot 1 \cdot 2^1 = 2A + 2B$. This system has solution $A = 2, B = -1$, so the specific solution is $a_n = 2 \cdot 2^n - n2^n$ or $a_n = 2^{n+1} - n2^n$.

9. Let S, T be sets. Prove that $S \Delta T \subseteq S \cup T$.

SOLUTION 1: Let $x \in S \Delta T$. Then $(x \in S \wedge x \notin T) \vee (x \notin S \wedge x \in T)$. We have two cases:
(Case $x \in S \wedge x \notin T$): By simplification, $x \in S$. By addition, $x \in S \vee x \in T$. Hence $x \in S \cup T$.
(Case $x \notin S \wedge x \in T$): By simplification, $x \in T$. By addition, $x \in S \vee s \in T$. Hence $x \in S \cup T$.
SOLUTION 2: We apply Thm 8.12, which states that $S \Delta T = (S \cup T) \setminus (S \cap T)$. We then apply the third problem on this exam, to conclude that $(S \cup T) \setminus (S \cap T) \subseteq (S \cup T)$. Combining these two gives the desired result.

10. Let R, S, T be sets. Prove that $R \times (S \cap T) \subseteq (R \times S) \cap (R \times T)$.

Let $x \in R \times (S \cap T)$. Then $x = (a, b)$, where $a \in R$ and $b \in S \cap T$. Hence $b \in S \wedge b \in T$. We will simplify this statement twice. By simplification the first time, $b \in S$, and hence $(a, b) \in R \times S$. By simplification the other way, $b \in T$, and hence $(a, b) \in R \times T$. Now, by conjunction, $((a, b) \in R \times S) \wedge ((a, b) \in R \times T)$. Hence, $(a, b) \in (R \times S) \cap (R \times T)$. Thus $x \in (R \times S) \cap (R \times T)$.