Spring 2010 Math 245-2 Exam 3 Solutions

One quarter of students scored 58-68, one quarter scored 68-76, one quarter scored 76-82, one quarter scored 83-98. In particular, the median was 76, the low was 58, and the high was 98.

Problem 1. Carefully define each of the following terms:

a. cardinal number
A cardinal number measures the size of some set.

b. ordinal number
An ordinal number measures sequential position, as compared to 'first'.

c. binary relation
A binary relation on A, B is a subset of $A \times B$.

d. union
The union of two sets is the set containing all the elements in either or both sets.

e. injective
A function is injective if every pair of distinct elements in the domain get mapped to distinct elements of the codomain.

Problem 2. For all sets A, B, C, prove that $A \cap B \cap C \subseteq A \cup C$.
Let $x \in A \cap B \cap C$. Then $(x \in A) \wedge (x \in B) \wedge (x \in C)$, by definition of $\cap$. Then, by conjunctive simplification, $x \in A$. Then, by disjunctive addition, $(x \in A) \vee (x \in C)$. Then, by definition of $\cup, x \in A \cup C$.

Problem 3. Let $A = \{a, b, c\}, B = \{b, c, d\}$. Find $\mathcal{P}(A) \cap \mathcal{P}(B)$.
The desired intersection is a set that contains as elements all C, such that $(C \subseteq A) \wedge (C \subseteq B)$. Hence, the answer is $\{\emptyset, \{b\}, \{c\}, \{b, c\}\}$.

Problem 4. For $A = \{a, b\}$, find a relation R on A that is not reflexive and not symmetric.
To make R not reflexive, it must not contain both (a, a) and (b, b). To make R not symmetric, it must contain either (a, b) or (b, a) but not both. Hence, six answers are possible; one example is $R = \{(a, a), (a, b)\}$.

Problem 5. Fix $A = \{1, 2, 3\}$. We define relation R on A via xRy if and only if $x + 3 < y^2$. Determine, with proof, whether R is a partial order.
We find $R = \{(1, 3), (2, 3), (3, 3)\}$. This is antisymmetric and transitive, but it is not reflexive since $(1, 1) \notin R$ (also $(2, 2) \notin R$), so R is not a partial order.

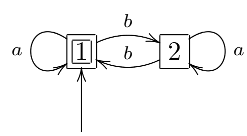
Problem 6. Fix $A = \{1, 2, 3\}$. We define relation R on A via xRy if and only if $x + 3 < y^2$. Determine, with proof, whether R is a function.
Since $R = \{(1, 3), (2, 3), (3, 3)\}$, we have $\mathbf{R(1) = 3, R(2) = 3, R(3) = 3}$. Hence R takes exactly one value for each element of the domain, so R is a function.

Problem 7. Let $f : X \to Y, g : Y \to Z$ be functions, with $g \circ f$ injective. Prove or disprove that f is injective.
SOLUTION 1: Let $a, b \in X$, with $f(a) = f(b)$. But then $g(f(a)) = g(f(b))$, so $(g \circ f)(a) = (g \circ f)(b)$. Since $g \circ f$ is injective, $a = b$. Hence f is injective.

SOLUTION 2: Argue by way of contradiction. Suppose f were not injective. Then there would be $a, b \in X$, with $a \neq b$, and $f(a) = f(b)$. But then $g(f(a)) = g(f(b))$, so $(g \circ f)(a) = (g \circ f)(b)$. Since $g \circ f$ is injective, this implies $a = b$, which is a contradiction. Hence f is injective.

Problem 8. Prove that any arrangement of five points in a unit square will have some pair of them within 0.75 of each other.
Divide the unit square into a 2×2 grid. By the pigeonhole principle, some 0.5×0.5 square must contain at least two points. The diagonal of this square is $\frac{1}{\sqrt{2}} < 0.75$, so this pair is within 0.75 of each other.

Problem 9. Find a finite-state automaton on alphabet $\{a, b\}$ that recognizes all strings with an even number of b's.

One solution is 

Problem 10. Solve the recurrence $a_n = a_{n-1} + 6a_{n-2}$, with initial conditions $a_0 = 4, a_1 = 7$.
The characteristic polynomial is $r^2 = r + 6, r^2 - r - 6 = 0, (r-3)(r+2) = 0$. Hence the general solution is $a_n = A3^n + B(-2)^n$. The initial conditions give $4 = a_0 = A + B$ and $7 = a_1 = 3A - 2B$. We solve these to get $A = 3, B = 1$. Hence the solution is $a_n = 3 \cdot 3^n + 1(-2)^n = 3^{n+1} + (-2)^n$.