Maximal-ly Surprising Triangle Functions

F. Ponomarenko and V. Ponomarenko

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Abstract - We look for functions that, evaluated symmetrically on the angles of a triangle and added, achieve their maximum at surprising values.

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1 Introduction

It is well-known (e.g. in [1]) that for any plane triangle with angles $\theta_1, \theta_2, \theta_3$, the function $\cos\theta_1+\cos\theta_2+\cos\theta_3 \leq \frac{3}{2}$, where $\frac{3}{2}$ is achieved with an equilateral triangle. This symmetric outcome is unsurprising, since the function is symmetric in the three angles. Considering instead $f(x) = \cos^2 x$, we again find the maximum occurs either for an equilateral triangle, or for a degenerate triangle.

One wonders if similar functions $f(x)$ on $[0, \pi]$ can exist, such that the triangle function $f(\theta_1)+f(\theta_2)+f(\theta_3)$ achieves its maximum on a surprising triangle, i.e., neither equilateral nor degenerate. Such a function $f(x)$ (also the triangle function $f(\theta_1) + f(\theta_2) + f(\theta_3)$) would be surprising at its maximum, i.e., maximal-ly surprising. The above examples show that $f(x) = \cos(x)$ and $f(x) = \cos^2(x)$ are not maximal-ly surprising[1], and perhaps it may seem that such functions do not exist. We will show that they do.

The natural approach to finding the absolute maximum would be with Lagrange multipliers. We seek to maximize $f(x) + f(y) + f(z)$, on the triangle formed by intersecting the plane $x + y + z = \pi$ with the first octant. We can find candidates using Lagrange multipliers on the interior of this triangle. Maxima on the boundary (i.e., if $xyz = 0$) would be unsurprising, as would be maxima at the center (i.e., $x = y = z = \frac{\pi}{3}$).

Here we will consider functions $f(x) = \cos^m(x)$, for all natural m. We will show that all odd $m > 1$ are maximal-ly surprising, while even m are not. Further, the triangle function at these surprising triangles approaches a limit as odd $m \to \infty$, and we will determine this limiting value, which is $2^{2/3} - 2^{-4/3}$. The first few maximal-ly surprising triangles are illustrated in Figure 1, below.

[1]Neither are the first or second powers of any of the six trigonometric functions.

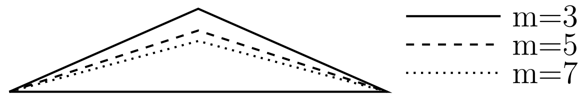

Figure 1: The maximal-ly surprising triangles for $f(x) = \cos^m(x)$ at $m = 3, 5, 7$.

First, we need a technical result, of some modest independent interest. It will allow us to rule out maxima that occur on triangles that are simultaneously nondegenerate and scalene, reducing the problem to isosceles triangles.

Proposition 1.1 *Let $m \in \mathbb{N}$, and set $g(x) = \cos^m(x)\sin(x)$. Suppose $\theta_1, \theta_2, \theta_3 \in [0, \pi]$ are distinct with $\theta_1 + \theta_2 + \theta_3 = \pi$ and $g(\theta_1) = g(\theta_2) = g(\theta_3)$. Then $\theta_1\theta_2\theta_3 = 0$.*

Proof. We first prove that $g(x)$ is unimodal on $[0, \pi/2]$. We calculate

$$\begin{aligned} g'(x) &= -m\cos^{m-1}(x)\sin^2(x) + cos^{m+1}(x) \\ &= \cos^{m-1}(x)(-m\sin^2(x) + \cos^2(x)) \\ &= \cos^{m-1}(x)(-m + (m+1)\cos^2(x)). \end{aligned}$$

Note that $\cos^{m-1}(x) > 0$ on $[0, \pi/2)$, while $-m + (m+1)\cos^2(x)$ is monotone decreasing from 1 down to $-m$. Hence $g(x)$ monotonically increases from $g(0) = 0$ to some maximum achieved at some $x^\star$, then monotonically decreases down to $g(\pi/2) = 0$. Also $g(x)$ is positive on $[0, \pi/2)$.

Next, we observe that if m is even then $g(\pi - x) = g(x)$, so $g(x)$ is unimodal and positive on $(\pi/2, \pi]$ as well. On the other hand, if m is odd then $g(\pi - x) = -g(x)$, so $-g(x)$ is unimodal and positive on $(\pi/2, \pi]$.

Now, suppose that m is even. Each horizontal line $y = M$ crosses the graph of g in 0, 2, 3, or 4 places. 3 crossings occurs only for $M = 0$, and 2 crossings occurs only if M is that unique maximum value, achieved once in $[0, \pi/2)$ and again in $(\pi/2, \pi]$. Suppose now that $g(\theta_1) = g(\theta_2) = g(\theta_3)$, for distinct $\theta_1, \theta_2, \theta_3$. If $M = 0$ then $\{\theta_1, \theta_2, \theta_3\} = \{0, \pi/2, \pi\}$ so $\theta_1\theta_2\theta_3 = 0$. Otherwise the three θ's are chosen from $\{x, \pi - x\} \cup \{y, \pi - y\}$ for some x, y. By the pigeonhole principle, two must be chosen from the same set, so without loss we have $\theta_1 = \pi - \theta_2$. But now $\pi = \theta_1 + \theta_2 + \theta_3 = (\pi - \theta_2) + \theta_2 + \theta_3$, so $\theta_3 = 0$ and hence $\theta_1\theta_2\theta_3 = 0$.

The case of m odd is simpler. Since $g(x)$ is unimodal and positive on $[0, \pi/2)$, and negative on $(\pi/2, \pi]$, $g(x) = M > 0$ has at most two distinct solutions, both in $[0, \pi/2)$. Similarly, $g(x) = M < 0$ has at most two distinct solutions, both in $(\pi/2, \pi]$. Hence $g(x) = M$ can have three distinct solutions only for $M = 0$, and again we have $\{\theta_1, \theta_2, \theta_3\} = \{0, \pi/2, \pi\}$ so $\theta_1\theta_2\theta_3 = 0$. □

2 Main Result

Now we are ready for the main result.

Theorem 2.1 *Let $m \in \mathbb{N}$ with $m \geq 2$, and set $f(x) = \cos^m(x)$. If m is even, then $f(x)$ is not maximal-ly surprising, i.e., there is no maximal-ly surprising triangle. If instead m is odd, then $f(x)$ is maximal-ly surprising. Further, there is exactly one maximal-ly surprising triangle, whose triangle sum $f(\theta_1)+f(\theta_2)+f(\theta_3)$ is within $\frac{0.88}{m-1}$ of the limiting value $2^{2/3} - 2^{-4/3} \approx 1.190550789$.*

Proof. We first consider the domain boundary, i.e., $\theta_1\theta_2\theta_3 = 0$. On that boundary, without loss of generality, $\theta_3 = 0$, so $\theta_2 = \pi - \theta_1$, and our triangle function is $f(0) + f(\theta_1)+f(\pi-\theta_1) = 1+\cos^m(\theta_1)+(-1)^m \cos^m(\theta_1)$. If m is even, then this is $1+2\cos^m(\theta_1)$, which has maximum 3, which is the global maximum since $f(\theta_i) \leq 1$ for $i = 1, 2, 3$. No interior maxima can beat this, so even m causes $f(x)$ to be not maximal-ly surprising.

We assume henceforth that m is odd. Then, the triangle function is constant, specifically 1, on the entire boundary. We will show that 1 is not maximal by finding a greater value in the interior.

We now turn our attention to the interior of the domain. Consider the Lagrangian $L = f(\theta_1) + f(\theta_2) + f(\theta_3) + \lambda(\theta_1 + \theta_2 + \theta_3 - \pi)$, with gradient

$$\nabla L = (u(\theta_1), u(\theta_2), u(\theta_3), \theta_1 + \theta_2 + \theta_3 - \pi),$$

where $u(\theta) = -m\cos^{m-1}(\theta)\sin(\theta)$. Setting $\nabla L = 0$ and rearranging, we find that we need both $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\cos^{m-1}(\theta_1)\sin(\theta_1) = \cos^{m-1}(\theta_2)\sin(\theta_2) = \cos^{m-1}(\theta_3)\sin(\theta_3)$. Applying Proposition 1.1, we find that if $\theta_1, \theta_2, \theta_3$ are distinct, then $\theta_1\theta_2\theta_3 = 0$, which is on the boundary.

Away from the boundary the θ_i are all positive. In this case Proposition 1.1 and the Lagrangian condition $\nabla L = 0$ say that the θ_i are not distinct at a maximum, so we may assume without loss of generality that our angles are $\theta = \theta_1 = \theta_2$ and $\pi - 2\theta = \theta_3$. Set $x = \cos\theta$, and note that $\cos\theta_3 = -\cos(2\theta) = -(2\cos^2(\theta)-1) = -2x^2+1$. So now we have reduced our problem to finding the maximum of the polynomial $g(x) = 2x^m+(-2x^2+1)^m$. Since $0 \leq \theta \leq \frac{\pi}{2}$, we have $x = \cos\theta \in (0, 1)$, where the endpoints are excluded since this would not be interior.

Now, $g'(x) = 2mx^{m-1} + (-4x)m(-2x^2 + 1)^{m-1} = 2mx(x^{m-2} - 2(-2x^2 + 1)^{m-1})$, whose zeroes in $(0, 1)$ coincide with the zeroes of $h(x) = x^{m-2} - 2(2x^2 - 1)^{m-1}$. Next, we will prove that $h(x)$ has two zeroes: $x = \frac{1}{2}$, and another $x \in (x_1, x_2) \subseteq I$, where $x_1 = 1 - \frac{\ln 2}{3(m-1)}$, $x_2 = 1 - \frac{\ln 2}{3m}$, and $I = \left(\frac{1}{\sqrt{2}}, 1\right]$.

First, we directly calculate $h(\frac{1}{2}) = \left(\frac{1}{2}\right)^{m-2} - 2\left(\frac{1}{2}\right)^{m-1} = 0$. Next, on I we have $2x^2 - 1 > 0$. Put $v(x) := (m-2)\ln x - (m-1)\ln(2x^2 - 1) - \ln 2$. Then $h(x) = 0$ if and only if $v(x) = 0$, and

$$v(1) = -\ln 2, \qquad v'(x) = \frac{m-2}{x} - \frac{4x(m-1)}{2x^2-1}, \qquad v''(x) = -\frac{m-2}{x^2} + (m-1)\frac{8x^2+4}{(2x^2-1)^2}.$$

For $x \in I$ the last term is greater than or equal to $12(m-1)$, since its minimum is at $x = 1$, hence

$$v''(x) \geq 12(m-1) - (m-2) = 11m - 10 > 0,$$

so v is convex on I. Moreover, for $x \in I$, $\frac{4x}{2x^2-1} \geq 3+\frac{1}{x}$, equivalently $6x^3-2x^2-3x-1 \leq 0$, so

$$v'(x) \leq \frac{m-2}{x} - (m-1)\left(3+\frac{1}{x}\right) = -\frac{1}{x} - 3(m-1) < 0.$$

Thus v is strictly decreasing on I. Next we will prove $v(x_1) > 0 > v(x_2)$. Since v is convex, its graph lies above its tangent at $x = 1$; hence for any $x \in I$,

$$v(x) \geq v(1) + v'(1)(x-1) = -\ln 2 + (3m-2)(1-x).$$

At $x = x_1$ we have $1 - x_1 = \frac{\ln 2}{3(m-1)}$, so

$$v(x_1) \geq -\ln 2 + \frac{(3m-2)\ln 2}{3(m-1)} = \frac{\ln 2}{3(m-1)} > 0,$$

hence $h(x_1) > 0$. Using convexity again, using a supporting line at x, for any $x \in I$, $v(1) \geq v(x) + v'(x)(1-x)$, which rearranges to $v(x) \leq v(1) - v'(x)(1-x)$. Because v' is increasing and $x_2 \in I$, we have $v'(x_2) \leq v'(1) = -(3m-2)$. Therefore

$$v(x_2) \leq -\ln 2 + (3m-2)(1-x_2) = -\ln 2 + \frac{(3m-2)\ln 2}{3m} = -\frac{2\ln 2}{3m} < 0,$$

and so $h(x_2) < 0$.

By the Intermediate Value Theorem, there is a zero of f in (x_1, x_2). Together with the zero at $x = \frac{1}{2} \in (0,1)$ and the fact that $v'(x) < 0$ on I (so $v(x) = 0$, equivalently $h(x) = 0$, has at most one solution there), the root in (x_1, x_2) is unique in I. Hence f has exactly two zeros in $(0,1)$.

Now, $x = \frac{1}{2}$ corresponds to an equilateral triangle, so this would not be maximal-ly surprising if it were maximal. However, it is not maximal, since the triangle function g evaluates to $3(\frac{1}{2})^m < 1$, even less than on the boundary.

Turning now to the other zero of $h(x)$, x', we set $c = \frac{\ln 2}{3}$ and compute

$$\begin{aligned}
f(x') &= 2(x')^m + \left(1 - 2(x')^2\right)^m \\
&\leq 2\left(1-\frac{c}{m}\right)^m + \left(1 - 2\left(1 - \frac{c}{m-1}\right)^2\right)^m \\
&= 2\left(1-\frac{c}{m}\right)^m - \left(1 - \frac{4c}{m-1} + \frac{2c^2}{(m-1)^2}\right)^m \\
&\leq 2\left(1-\frac{c}{m}\right)^m - \left(1 - \frac{4c}{m-1}\right)^m \\
&= 2\left(1-\frac{c}{m}\right)^m - \left(1 - \frac{4c}{m-1}\right)^{m-1}\left(1 - \frac{4c}{m-1}\right) \\
&\leq 2e^{-c} - e^{-4c-\frac{c^2}{2(m-1-c)}}\left(1 - \frac{4c}{m-1}\right),
\end{aligned}$$

where in the last step we used the standard bounds $e^{-c-\frac{c^2}{2(x-c)}} \leq (1-\frac{c}{x})^x \leq e^{-c}$ (valid for $x > c$). In the other direction, we have

$$\begin{aligned}
f(x') &= 2(x')^m + \left(1-2(x')^2\right)^m \\
&\geq 2\left(1-\frac{c}{m-1}\right)^m + \left(1-2\left(1-\frac{c}{m}\right)^2\right)^m \\
&= 2\left(1-\frac{c}{m-1}\right)^{m-1}\left(1-\frac{c}{m-1}\right) - \left(1-\frac{4c}{m}+\frac{2c^2}{m^2}\right)^m \\
&\geq 2e^{-c-\frac{c^2}{2(m-1-c)}}\left(1-\frac{c}{m-1}\right) - e^{-4c+\frac{2c^2}{m}}.
\end{aligned}$$

Note that as $m \to \infty$, both upper and lower bounds approach $2e^{-c}-e^{-4c} = 2^{2/3}-2^{-4/3}$. It only remains to estimate convergence rate.

We will bound the gap between the upper and lower bounds, which we call $E(m)$. We have

$$E(m) = 2e^{-c}\left(1-\left(1-\frac{c}{m-1}\right)e^{-\frac{c^2}{2(m-1-c)}}\right) + e^{-4c}\left(e^{\frac{2c^2}{m}} - \left(1-\frac{4c}{m-1}\right)e^{-\frac{c^2}{2(m-1-c)}}\right)$$

Set $x_1 = \frac{c^2}{2(m-1-c)}$ and $x_2 = \frac{2c^2}{m}$. Now, $x_1 = \frac{c^2}{2(m-1)}\left(1-\frac{c}{m-1}\right)^{-1}$. Since $\left(1-\frac{c}{m-1}\right)^{-1} \leq \left(1-\frac{\log 2}{6}\right)^{-1} \leq 1.14$, so $x_1 \leq 1.14\frac{c^2}{2(m-1)}$. Also, $x_2 \leq \frac{2c^2}{m-1}$.

Since $m \geq 3$ and $c = \frac{\ln 2}{3}$ we have $x_1, x_2 \in (0, 0.05)$. We multiply $1-e^{-x_1} \leq x_1$ on both sides by $1-\frac{c}{m-1}$ and rearrange to find

$$1-\left(1-\frac{c}{m-1}\right)e^{-x_1} \leq \frac{c}{m-1}(1-x)+x_1 \leq \frac{c}{m-1}+1.14\frac{c^2}{2(m-1)}.$$

We now multiply $1-e^{-x_1} \leq x_1$ on both sides by $1-\frac{4c}{m-1}$ and rearrange to find

$$1-\left(1-\frac{4c}{m-1}\right)e^{-x_1} \leq \frac{4c}{m-1}(1-x_1)+x_1,$$

and therefore

$$\begin{aligned}
e^{x_2}-\left(1-\frac{4c}{m-1}\right)e^{-x_1} &\leq \frac{4c}{m-1}(1-x_1)+x_1+(e^{x_2}-1) \\
&\leq \frac{4c}{m-1}+1.14\frac{c^2}{2(m-1)}+2x_2 \\
&\leq \frac{4c}{m-1}+1.14\frac{c^2}{2(m-1)}+\frac{4c^2}{m-1}.
\end{aligned}$$

Putting it all together, we find

$$E(m) \leq \frac{2e^{-c}(c+0.57c^2)+e^{-4c}(4c+0.57c^2+4c^2)}{m-1} \leq \frac{0.88}{m-1}.$$

□

We close by inviting the reader to look for other maximal-ly surprising triangle functions, which may provide other magical values, like $2^{2/3}-2^{-4/3}$.

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Felix Ponomarenko
High Tech High International
E-mail: `felixsponomarenko@gmail.com`

Vadim Ponomarenko
Department of Mathematics and Statistics
San Diego State University
5500 Campanile Dr.
San Diego, CA 92182-7720
E-mail: `vponomarenko@sdsu.edu`

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