Minimal Zero Sequences of Finite Cyclic Groups

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Let $MZS(G, k)$ denote the set of minimal zero sequences of finite Abelian group G. In this paper we investigate the structure of the elements of this set, and the cardinality of the set itself. We do this for the class of groups $G = \mathbb{Z}_n$ for k both small ($k \leq 4$) and large ($k > \frac{2n}{3}$).

Key Words: Zero-sum problems, minimal zero sequence

1. INTRODUCTION

Let G be a finite Abelian group and $X = \{x_1, x_2, \ldots, x_k\}$ a multiset chosen from G. This unordered collection of not necessarily distinct elements of G is traditionally called a *sequence*. We say the *length* of X is k. If $x_1 + x_2 + \cdots + x_k = 0$ (in G), then X is called a *zero-sequence*. We denote the set of all zero sequences of G by $ZS(G)$. If X is in $ZS(G)$ but no proper subsequence of X is in $ZS(G)$, then X is called a *minimal zero sequence*. We denote the set of all minimal zero sequences of G of length k by $MZS(G, k)$, and the set of all minimal zero sequences of G of any length by $MZS(G)$. The maximum k for which $MZS(G, k)$ is nonempty is the well-known Davenport constant of G.

Notice that $\mathrm{Aut}(G)$ acts on $ZS(G)$, on $MZS(G)$, and on $MZS(G, k)$, inducing equivalence relations on these sets. We denote by $E(X)$ the set of sequences equivalent to sequence X, as induced in this manner.

We express G canonically as $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, with $n_1 | n_2 | \cdots | n_r$. We say that zero sequence X is *basic* if $E(X)$ contains a zero sequence whose sum in coordinate i is at most n_i (for $1 \leq i \leq r$), when the sum is viewed as an integer. To avoid confusion, henceforth the symbol '+' shall denote addition as integers, and the symbol $\sum X$ shall denote the sum of the elements of X as integers. If G is cyclic, that is its rank $r = 1$, then all basic zero sequences are minimal.

If every element of $MZS(G,k)$ is basic, we say that (G,k) is a *basic pair*, otherwise it is a *non-basic pair*. Chapman, Freeze, and Smith [2] have shown that $(\mathbb{Z}_n, 5)$ is a non-basic pair for all $n \neq 2,3,4,5,7$; further, for these five values of n, $(\mathbb{Z}_n, k)$ is a basic pair for all k. This left open the question of which $(\mathbb{Z}_n, k)$ are basic pairs.

We offer a partial answer to this question, for all $n \geq 5$ and both very small and large k. We show in Theorems 2.1 and 3.4 that $(\mathbb{Z}_n, k)$ is a basic pair for $k > \frac{2n}{3}$ and $k \leq 3$; whereas $(\mathbb{Z}_n, 4)$ is a non-basic pair if $\gcd(n, 6) \neq 1$.

As an application, we count the number of minimal zero sequences of length greater than $\frac{2n}{3}$.

2. SHORT MINIMAL ZERO SEQUENCES

We first consider the question of whether $(\mathbb{Z}_n, k)$ is a basic pair for $n \leq 4$. We have evidence to support the converse of the second part of the theorem; that is, we believe that if $\gcd(n, 6) = 1$, then $(\mathbb{Z}_n, 4)$ is a basic pair. This has been verified computationally for $n \leq 1000$.

THEOREM 2.1. *Let $n \geq 5$. Then $(\mathbb{Z}_n, k)$ is a basic pair for $k = 1, 2, 3$. If $\gcd(n, 6) \neq 1$, then $(\mathbb{Z}_n, 4)$ is a non-basic pair.*

Proof. The only element of $MZS(\mathbb{Z}_n, 1)$ is $\{0\}$, which is basic. Let $X = \{a, b\} \in MZS(\mathbb{Z}_n, 2)$. It has $a < n$ and $b < n$, and hence $a + b < 2n$, so X is basic. Suppose that $X = \{a, b, c\} \in MZS(\mathbb{Z}_n, 3)$ were non-basic. Then $a+b+c > n$, but $a+b+c < 3n$, so $a+b+c = 2n$. Now, $\phi(y) = n-y$ is an automorphism on $MZS(\mathbb{Z}_n, 3)$, and $\phi(X) = \{n-a, n-b, n-c\}$ has $\sum \phi(X) = (n-a) + (n-b) + (n-c) = 3n - (a+b+c) = 3n - 2n = n$. Hence X is, in fact, basic.

Suppose now that n is even, so $n = 2m$. We will now show that $X = \{1, m, m+1, 2m-2\}$ is not basic, and hence that $(\mathbb{Z}_{2m}, 4)$ is a non-basic pair. First, X sums to a multiple of n, but no proper subset does, hence X is a minimal zero sequence. Now, let ϕ be any automorphism of $\mathbb{Z}_{2m}$. We must have $\phi(y) = ky$, for k some positive odd integer, different from m, less than n. We see that $\phi(X) = \{k, km, km+k, k(2m-2)\}$. Reducing modulo n, we see that $\phi(X) = \begin{cases} \{k, m, m+k, 2m-2k\} & \text{if } k < m, \\ \{k, m, k-m, 4m-2k\} & \text{if } k > m. \end{cases}$
In both cases we have $\sum \phi(X) = 2n$. Hence, X is not basic if n is even.

Now suppose that $3|n$; that is, $n = 3m$. We will now show that $X = \{1, m+1, 2m+1, 3m-3\}$ is not basic, and hence that $(\mathbb{Z}_n, 4)$ is a non-basic pair. First, X sums to a multiple of n, but no proper subset does, hence X is a minimal zero sequence. Now, let ϕ be any automorphism of $\mathbb{Z}_n$. We must

have $\phi(y) = ky$, for k some positive integer, less than n, relatively prime to n. We have $\phi(X) = \{k, km + k, 2km + k, 3km - 3k\}$. We next note that $\{km+k, 2km+k\}$ are congruent (modulo n) to $\{m+k, 2m+k\}$ in some order, depending on whether $k \equiv 1$ or $k \equiv 2$ (modulo 3). We can now reduce modulo n, and find $\phi(X) = \begin{cases} \{k, m+k, 2m+k, 3m-3k\} & \text{if } k < m, \\ \{k, m+k, k-m, 6m-3k\} & \text{if } m < k < 2m, \\ \{k, k-2m, k-m, 9m-3k\} & \text{if } 2m < k. \end{cases}$
In all three cases we have $\sum \phi(X) = 2n$. Hence, X is not basic if $3|n$. ▮

3. LONG MINIMAL ZERO SEQUENCES

We now consider minimal zero sequences in $\mathbb{Z}_n$, long relative to the maximal possible length (namely n). We begin with some structure theorems, and ultimately show that $(\mathbb{Z}_n, k)$ is a basic pair for all $k > \frac{3n-3}{4}$.

We state a theorem that was first proved in [1], was rediscovered in [7], and restated in various forms in [6, 8].

THEOREM 3.1. *Let $k > \frac{n+3}{2}$, and let $X \in MZS(\mathbb{Z}_n, k)$. Then there is some element $a \in \mathbb{Z}_n$ that appears in X at least $2k - n$ times.*

With a stronger restriction on k, we can get a bit more. This next result has a stronger hypothesis and conclusion than a similar one found in [5]. It has previously appeared in [4], with a substantially different proof.

THEOREM 3.2. *Let $k > max(\frac{n+3}{2}, \frac{2n}{3})$, and let $X \in MZS(\mathbb{Z}_n, k)$. Then there is some element $a \in \mathbb{Z}_n$ that appears in X at least $2k-n$ times, whose order is n (in $\mathbb{Z}_n$).*

Proof. Applying Theorem 3.1, we write $X = \{a^m, b_1, b_2, \ldots, b_j\}$ (where m is the multiplicity of a), with $m \geq 2k - n$ and $m + j = k$.

Now, suppose that the order of a were less than n. Then, we can write $a = a'd$ and $n = n'd$, where $\gcd(a', n') = 1$ and $d \geq 2$. However, if $d \geq 3$, we have $n' \leq \frac{n}{3} < m$. Hence X contains n' copies of a, whose sum is $n'a = n'da' = na'$. But this is a proper zero-sum, which is forbidden. Therefore, we must have $d = 2$, n even (since $d|n$), and $m < \frac{n}{2}$ (since a^m is not a zero subsequence). The remainder of the proof develops a contradiction in these circumstances.

We now show that there is an automorphism ϕ of G with $\phi(a) = 2$. Because $\gcd(a', n') = 1$, there is some integer w with $wa' \equiv 1$ modulo n'. If w is odd, then $\gcd(w, n) = 1$ and $\phi(x) = wx$ is the desired automorphism. If w is even, then n' must be odd. In this case, $(w + n')a' \equiv 1$ modulo n'. We have $w + n'$ odd, so $\gcd(w + n', n) = 1$ and therefore $\phi(x) = (w + n')x$

is the desired automorphism. Henceforth we will assume without loss that $a = 2$.

We now consider the odd elements of X. We pair them arbitrarily and take the residue modulo n. The result is $X' = \{2^m, c_1, c_2, \dots, c_{j'}\}$, where some $c_{i'}$ are equal to an even b_i, while others are equal to the reduced sum of two odd elements of X. This is still a minimal zero sequence, and all of its terms are even. Further, we have $j' \geq \frac{j}{2}$. Note that $m + j = k > \frac{2n}{3}$, and hence $j' \geq \frac{j}{2} > \frac{n}{3} - \frac{m}{2} > \frac{n}{3} - \frac{n}{4} + \frac{1}{2} = \frac{n}{12} + \frac{1}{2}$. Therefore, in particular, $j' \geq 2$. Now we will show that any proper subsequence of $\{c_1, c_2, \dots, c_{j'}\}$ has sum at most $n - 2m - 2$, by induction on the cardinality of the subsequence. For the base case, observe that each singleton c_i must have $c_i \leq n - 2m - 2$, as otherwise X' would not be a minimal zero sequence. Now, let S be a proper subsequence. Write $S = S_1 \cup S_2$, the disjoint union of two nonempty subsequences. By the inductive hypothesis, $\sum S_1 \leq n - 2m - 2$ and $\sum S_2 \leq n - 2m - 2$. Adding, we get $\sum S = \sum S_1 + \sum S_2 \leq 2n - 4m - 4 \leq 2n - \frac{4n}{3} - 4 = \frac{2n}{3} - 4 < n$. We have $\sum S$ even, but because S is a proper subsequence, we must not have $\sum S \in [n - 2m, n]$. Therefore $\sum S \leq n - 2m - 2$. Finally, we note that $(c_1 + c_2 + \cdots + c_{j'-1}) + c_{j'} \leq n - 2m - 2 + n - 2m - 2 \leq \frac{2n}{3} - 4 < n$. Therefore, because X' is a minimal zero sequence, we must have $2m + c_1 + c_2 + \cdots + c_{j'} = n$. However, each c_i is even, so we therefore have the chain of inequalities $n = \frac{n}{3} + \frac{2n}{3} < m + k = 2m + j \leq 2m + 2j' \leq 2m + c_1 + \cdots + c_{j'} = n$. This is a contradiction. ▮

Corollary 3.1. *Let $n \geq 10, k > \frac{2n}{3}$, and let $X \in MZS(\mathbb{Z}_n, k)$. Then there is some element $a \in \mathbb{Z}_n$ that appears in X more than $\frac{k}{2}$ times, whose order is n (in $\mathbb{Z}_n$).*

Proof. The condition $n \geq 10$ ensures that $\frac{2n}{3} \geq \frac{n+3}{2}$, so that the conditions of Theorem 3.2 are met. As before, we write $X = \{a^m, b_1, b_2, \dots, b_j\}$. Since $k > \frac{2n}{3}$, we must have $m > 2(\frac{2n}{3}) - n = \frac{n}{3}$. We also have $m + j = k$, and hence $m \geq 2k - n = (m + j) + k - n$. Rearranging, we get $j \leq n - k < \frac{n}{3}$. Combining these two facts, we get $j < \frac{n}{3} < m$, and hence $m > \frac{k}{2}$. ▮

This allows us to conclude that all sufficiently long minimal zero sequences of $\mathbb{Z}_n$ are basic.

Theorem 3.3. *Let $n \geq 10, k > \frac{3n-3}{4}$. Then $MZS(\mathbb{Z}_n, k)$ is a basic pair.*

Proof. Let $Y \in MZS(\mathbb{Z}_n, k)$. By Theorem 3.2 and Corollary 3.1, there is some element $y \in Y$, of order n, that appears at least $2k - n$ times. Let $\phi \in Aut(\mathbb{Z}_n)$ be such that $\phi(y) = 1$. Let $X = \phi(Y)$. We will show that $\sum X = n$, which proves the theorem. Write $X = \{1^m, x_1, x_2, \dots, x_j\}$, where $m \geq 2k - n$, $m + j = k$, and each $x_i > 1$. First, note that if $j = 1$ then $\sum X = m + x_j < m + n < 2n$, so $\sum X = n$. Otherwise, $j > 1$ and we see that each $x_i \leq n - m - 1$, since otherwise X would properly contain a zero sequence. Now, $x_1 < n - m$, but $x_1 + x_2 + \cdots + x_j \geq n - m$. Let w be such that $x_1 + x_2 + \cdots + x_{w-1} < n - m$, but $x_1 + x_2 + \cdots + x_w \geq n - m$. If $w = j$, then because $x_w < n$, we have $x_1 + x_2 + \cdots + x_w = n - m$ and hence $\sum X = n$. Otherwise, $x_1 + x_2 + \cdots + x_w \geq n + 1$ because X is a minimal zero sequence. Subtracting, we get $x_w \geq m + 2$. However, $n - m - 1 \geq x_w \geq m + 2$. Rearranging, we get $m \leq \frac{n-3}{2}$. But also $m \geq 2k - n > 2\frac{3n-3}{4} - n = \frac{n-3}{2}$. This is impossible, and hence $w = j$ and thus $\sum X = n$. ∎

It has come to our attention that a stronger result, with a different proof, has been published in [4]:

Theorem 3.4. *Let $n \geq 10, k > \frac{2n}{3}$. Then $MZS(\mathbb{Z}_n, k)$ is a basic pair.*

4. COUNTING MINIMAL ZERO SEQUENCES

The cardinality of $MZS(\mathbb{Z}_n, k)$ has already been computed for small k, in [3], as follows.

Theorem 4.1. *$|MZS(\mathbb{Z}_n, 2)| = \lfloor \frac{n}{2} \rfloor$. $|MZS(\mathbb{Z}_n, 3)| = \frac{1}{6}(n^2 - \alpha)$, where α is given by:*

$(n \mod 6) \equiv$	0	1	2	3	4	5
$\alpha =$	0	1	4	-3	4	1

We can find $|MZS(\mathbb{Z}_n, k)|$ for large k with the results of Section 3. For this purpose, we need the following structure theorem.

Theorem 4.2. *Let $n \geq 10, k > \frac{2n}{3}$, and let $X \in MZS(\mathbb{Z}_n, k)$ be basic. Then there is exactly one $Y \in E(X)$ with $\sum Y = n$.*

Proof. As X is basic, so at least one such Y exists. Suppose Y has i terms of 1, and the remaining $k - i$ terms are not. Hence $n = \sum Y \geq i + 2(k - i) = 2k - i$. Hence $i \geq 2k - n > 2k - \frac{3}{2}k = \frac{k}{2}$. Hence over half of the terms of Y are 1. Suppose that there are $Y, Y' \in E(X)$ with $\sum Y = \sum Y' = n$. Let $\phi \in Aut(\mathbb{Z}_n)$ with $\phi(Y) = Y'$. By the previous, 1 appears in each more than $\frac{k}{2}$ times. Both 1, $\phi(1)$ appear more than $|Y'|/2$ times in Y', but there are not enough elements in Y' for these to be different. Hence $\phi(1) = 1$, and therefore ϕ is the identity and $Y = Y'$. ∎

We are now ready to count all minimal zero sequences of sufficiently large length. Computational evidence suggests that the condition $k > \frac{2n}{3}$ can be improved to $k \geq \frac{n+4}{2}$.

THEOREM 4.3. *Let* $n \geq 10, k > \frac{2n}{3}$. *Then* $|MZS(\mathbb{Z}_n, k)| = \phi(n)p_k(n)$, *where* ϕ *is Euler's totient function and* $p_k(n)$ *denotes the number of partitions of* n *into* k *parts.*

Proof. By Theorem 3.4, every minimal zero sequence is basic. Therefore, each equivalence class induced by $Aut(\mathbb{Z}_n)$ includes an element whose sum is n. By Theorem 4.2, each equivalence class contains exactly one element whose sum is n. It is clear that the set of minimal zero sequences whose sum is n is exactly the set of partitions of n into k parts. There are therefore $p_k(n)$ equivalence classes. The cardinality of each equivalence class is $|Aut(\mathbb{Z}_n)| = \phi(n)$. ∎

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