Mimeomatroids

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A mimeomatroid is a matroid union of a matroid with itself. We develop several properties of mimeomatroids, including a generalization of Rado's theorem, and prove a weakened version of a matroid conjecture by Rota[2].

Key Words: mimeomatroid, matroid

1. INTRODUCTION

A mimeomatroid is constructed by taking the matroid union of a matroid with itself. This simple operation can be used to find a generalization of Rado's theorem – to address the question of when a family of subsets of a matroid has multiple transversals, each independent in the matroid. It can also partially confirm a matroid conjecture of Rota[2] concerning whether the elements of n bases $B_1, B_2, \ldots B_n$ of a rank-n matroid can always be repartitioned into other bases $B'_1, B'_2, \ldots, B'_n$ so that $|B_i \cap B'_j| = 1$ for all i, j.

We begin by recalling several key notions involving matroid union and matroid duality, an intersection theorem of Edmonds, and Rado's theorem. For more background on these topics, see [3, 6].

Let M_1 and M_2 be matroids on ground set E having independent sets I_1 and I_2, respectively. Then $I = \{I_1 \cup I_2 : I_1 \in I_1, I_2 \in I_2\}$ is the set of independent sets of a matroid on E, $M_1 \bigvee M_2$. Furthermore, for $X \subseteq E$, the rank of X in $M_1 \bigvee M_2$ is $\min\{|X \setminus Y| + r_1(Y) + r_2(Y) : Y \subseteq X\}$.

We now recall two classical theorems of matroid theory.

THEOREM 1.1 (Whitney[7]). *Let M be a matroid with ground set E. Let $\mathbf{B}^\star = \{E \setminus B : B$ is a basis of $M\}$. These are the bases of a matroid*

* Portions of this work comprise part of the author's PhD thesis written under the direction of Richard Brualdi at the University of Wisconsin at Madison.

$M^\star(E, \mathbf{B}^\star)$, called the dual matroid to M. Let $r^\star$ denote the rank function of $M^\star$. Then, for all subsets X of the ground set E, $r^\star(X) = |X| - r(E) + r(E - X)$.

THEOREM 1.2 (Edmonds[1]). *Let M_1 and M_2 be matroids with a common ground set E, and rank functions r_1 and r_2 respectively. Then there is a k-element subset of E that is independent in both M_1 and M_2 if and only if, for all subsets X of E, $r_1(X) + r_2(E - X) \geq k$.*

Let $(A_j : j \in J)$ be a family of subsets of a set E. We recall that a transversal of this family is defined to be a set of $|J|$ distinct elements $\{e_1, e_2, \dots, e_{|J|}\} \subseteq E$ with $e_j \in A_j$ for each $j \in J$. We now recall Rado's classical theorem on transversals.

THEOREM 1.3 (Rado[5]). *Let $(A_j : j \in J)$ be a family of subsets of a set E. Let M be a matroid on E having rank function r. Then $(A_j : j \in J)$ has a transversal that is independent in M if and only if, for all $K \subseteq J$, $r(\bigcup_{j \in K} A_j) \geq |K|$.*

In the following section we define mimeomatroids and derive some of their properties. This includes a generalization of Rado's theorem and several relationships between the rank functions of mimeomatroids of different multiplicities. In the final section we use those properties to confirm a weaker version of a conjecture of Rota[2].

2. MIMEOMATROIDS

Let M be a matroid, and let $d \in \mathbf{Z}^{\geq 1}$. Consider the matroid union $\bigvee_{i=1}^{d} M_i$, where $M_i = M$ for $1 \leq i \leq d$. We call this matroid a *mimeomatroid* of multiplicity d, with rank function $r_d()$, and abbreviate $\bigvee_{i=1}^{d} M_i$ as $\overset{d}{\bigvee} M$. Observe that if $d_1 \geq d_2$, then $r_{d_1}(X) \geq r_{d_2}(X)$ for any $X \subseteq E$.

The rank functions of mimeomatroids of various multiplicities are related by the following theorem.

THEOREM 2.1. *Let M be a matroid with ground set E, and let $X \subseteq E$. Let $a \geq b \geq c \geq d \geq 1$. Then $\frac{r_a(X)}{a} \leq \frac{r_b(X)+r_c(X)}{b+c} \leq \frac{r_d(X)}{d}$.*

Proof. Let $I_a^1 \dot\cup I_a^2 \dot\cup \cdots \dot\cup I_a^a \subseteq X$, $I_b^1 \dot\cup I_b^2 \dot\cup \cdots \dot\cup I_b^b \subseteq X$, $I_c^1 \dot\cup I_c^2 \dot\cup \cdots \dot\cup I_c^c \subseteq X$, $I_d^1 \dot\cup I_d^2 \dot\cup \cdots \dot\cup I_d^d \subseteq X$ each be maximal and independent disjoint unions of independent sets in M. Rearrange superscripts if necessary to have $|I_\alpha^1| \geq |I_\alpha^2| \geq \cdots \geq |I_\alpha^\alpha|$ for $\alpha = a, b, c, d$. We have $r_a(X) = |I_a^1 \dot\cup I_a^2 \dot\cup \cdots \dot\cup I_a^a| \leq |I_a^1 \dot\cup I_a^2 \dot\cup \cdots \dot\cup I_a^d| + (a-d)|I_a^d| \leq r_d(X) + (a-d)|I_a^d| \leq r_d(X) + (a-d)\frac{r_d(X)}{d} = \frac{a}{d} r_d(X)$. Similarly, we have $r_a(X) \leq \frac{a}{b} r_b(X)$, $r_a(X) \leq \frac{a}{c} r_c(X)$, $r_b(X) \leq \frac{b}{d} r_d(X)$, and $r_c(X) \leq \frac{c}{d} r_d(X)$. Combining these inequalities, we get $(\frac{b}{a} + \frac{c}{a}) r_a(X) \leq r_b(X) + r_c(X) \leq (\frac{b}{d} + \frac{c}{d}) r_d(X)$, from which the theorem follows. ▌

This result is in some sense best possible, since for the matroid $U_{0,n}$ we have $0 = \frac{r_a(X)}{a} = \frac{r_b(X)+r_c(X)}{b+c} = \frac{r_d(X)}{d}$ for any $X \subseteq E$.

The following are several natural corollaries of Theorem 2.1.

COROLLARY 2.1. *Let M be a matroid on ground set E, and let $X \subseteq E$. Then for a mimeomatroid of any multiplicity d, we must have $dr(X) \geq r_d(X) \geq r(X)$.*

COROLLARY 2.2. *Let M be a matroid on ground set E, and let $X \subseteq E$ be independent in $\overset{d}{\bigvee} M$ for some d. Then for any $1 \leq i \leq d$, we must have $r_i(X) + r_{d-i}(X) \geq |X|$.*

The following is a generalization of Rado's Theorem (1.3) to mimeomatroids; Rado's Theorem corresponds to $d = 1$.

THEOREM 2.2. *Let $(A_j : j \in J)$ be a family of subsets of a set E. Let M be a matroid on E. Let $d \in \mathbf{Z}^{\geq 1}$. Then, $(A_j : j \in J)$ has d transversals $\{e_j^i : e_j^i \in A_j, 1 \leq i \leq d, j \in J\}$ independent in the mimeomatroid $\overset{d}{\bigvee} M$ if and only if, for all $K \subseteq J, r_d(\bigcup_{j \in K} A_j) \geq d|K|$.*

Proof. First, we assume that $(A_j : j \in J)$ has d transversals $\{e_j^i : e_j^i \in A_j, 1 \leq i \leq d, j \in J\}$ independent in $\overset{d}{\bigvee} M$. Let $K \subseteq J$. Set $X = \{e_j^i : 1 \leq i \leq d, j \in K\}$. By construction, we have $X \subseteq \bigcup_{j \in K} A_j$ and X is independent in $\overset{d}{\bigvee} M$. Hence, we must have $r_d(\bigcup_{j \in K} A_j) \geq r_d(X) \geq |X| = d|K|$.

Now, we assume that for all $K \subseteq J, r_d(\bigcup_{j \in K} A_j) \geq d|K|$. For convenience, set $D = \{1, 2, \ldots, d\}$. Consider the family of subsets $(A_j^i : i \in D, j \in J)$, with $A_j^i = A_j$.

If we take any $K' \subseteq D \times J$, then we must have $r_d(\bigcup_{(i,j)\in K'} A_j^i) \geq |K'|$. This is because we can set $K \subseteq J$ minimal so that $K' \subseteq D \times K$, and get $r_d(\bigcup_{(i,j)\in K'} A_j^i) = r_d(\bigcup_{j\in K} A_j) \geq d|K| \geq |K'|$.

We now observe that $\overset{d}{\bigvee} M$ is a matroid on E with rank function r_d, that $(A_j^i : i \in D, j \in J)$ is a family of subsets of E, and that for all $K' \subseteq D \times J$, we have $r_d(\bigcup_{(i,j)\in K'} A_j^i) \geq |K'|$. By Theorem 1.3, there must be a transversal of $(A_j^i : i \in D, j \in J)$ that is independent in $\overset{d}{\bigvee} M$. This transversal is also d transversals of $(A_j : j \in J)$, from which the theorem follows. ∎

Observe that the $\{e_j^i : e_j^i \in A_j, 1 \leq i \leq d, j \in J\}$ provided by the theorem can be partitioned into d sets, each independent in M, whose union is d transversals of $(A_j : j \in J)$. This condition is weaker than having d transversals, each independent in M.

3. APPLICATION

Let M be a matroid of rank n on ground set E. Suppose $B_1, B_2, \ldots, B_n$ are pairwise nonintersecting bases of M. Rota conjectured in [2] that there always exists an $n \times n$ matrix A, whose jth column consists of the elements of B_j, ordered in such a way that the rows of A are bases as well.

We now confirm a weaker version of this conjecture, namely that there always exists an $n \times n$ matrix A whose jth column consists of the elements of B_j, and that the first d rows are a disjoint union of d bases, for each $1 \leq d \leq n$. An entirely different approach to this problem using jump systems can be found in [4].

THEOREM 3.1. *Let M be a matroid of rank n on ground set E. Let $B_1, B_2, \ldots, B_n$ be pairwise nonintersecting bases of M. There exists an $n \times n$ matrix A whose jth column consists of the elements of B_j and with the first d rows a basis of $\overset{d}{\bigvee} M$, for each $1 \leq d \leq n$.*

Proof. We proceed by induction down from n. The base case of $d = n$ is trivial. We now assume as given $S \subseteq E$, a basis of $\overset{d+1}{\bigvee} M$ with $|S \cap B_j| = d + 1$ for $1 \leq j \leq n$. Set $J = \{1, 2, \ldots, n\}$. Let $K \subseteq J$. By Theorem 2.1 we have $r_d(\bigcup_{j\in K} B_j \cap S) \geq \frac{d}{d+1} r_{d+1}(\bigcup_{j\in K} B_j \cap S) = \frac{d}{d+1} |\bigcup_{j\in K} B_j \cap S| = \frac{d}{d+1}|K|(d+1) = d|K|$. By Theorem 1.3, we must therefore have d

transversals $\{e_j^i : e_j^i \in B_j \cap S, 1 \leq i \leq d, j \in J\}$ that are independent in $\overset{d}{\bigvee} M$. If we set $T = \bigcup e_j^i$, then $|T| = dn$ and hence T is a basis of $\overset{d}{\bigvee} M$. ∎

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