Winning Moves in Fibonacci Nim

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http://www-rohan.sdsu.edu/~vadim/fnim.pdf

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Fibonacci Nim

Start with one pile of tokens, arbitrary size.

Two players alternate removing tokens.

Remove last token to win.

Special rule: If opponent just removed k tokens, you remove any integer in $[1, 2k]$.

First turn rule: Must move, can't instantly win.

Zeckendorf Representation

Zeckendorf had a proof in 1939

Lekkerkerker published in 1955

Zeckendorf published in 1972

"Zeckendorf representation" of positive integers

$F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \dots$

Each positive integer may be written as the sum of nonconsecutive Fibonacci numbers, uniquely.

e.g. $10 = F_6 + F_3,\ 11 = F_6 + F_4,\ 12 = F_6 + F_4 + F_2$

Schwenk's Strategy

Schwenk published in 1970

Strategy:
Express the number of tokens current remaining in Zeckendorf representation. Take the smallest.

If you can: forced win.
If you can't: opponent has forced win.

e.g. $17 = F_7 + F_4 + F_2 \to$ take F_2

Our Question

Suppose you can take the smallest Fibonacci number in the Zeckendorf representation. Hence you can force a win.

Are there any moves other than Schwenk's strategy that will force a win?

Tails

Given a Zeckendorf representation of the number of tokens remaining, the tails are the consecutive end terms.

$$17 = \underbrace{F_7 + \underbrace{F_4 + \underbrace{F_2}}}$$ has tails F_2, $F_4 + F_2$, $F_7 + F_4 + F_2$

$$12 = \underbrace{F_6 + \underbrace{F_4 + \underbrace{F_2}}}$$ has tails F_2, $F_4 + F_2$, $F_6 + F_4 + F_2$

Thm 1: If you don't take a tail, you lose.

Which Tails?

$17 = F_7 + F_4 + F_2$ has tails F_2, $F_4 + F_2$, $F_7 + F_4 + F_2$

$12 = F_6 + F_4 + F_2$ has tails F_2, $F_4 + F_2$, $F_6 + F_4 + F_2$

Thm 2: If you do take a tail, you can force a win, UNLESS your tail lets your opponent take the next Fib. number.

Ex.1: $F_2 + F_4 = 4$, $2 \cdot 4 = 8 < 13 = F_7$. So $F_2 + F_4$ wins.
Ex.2: $F_6 = 8$, so $F_2 + F_4$ does not win.

How Likely is Opponent to Accidentally Play Right?

Worst case opening scenario:
$F_2 + F_4 + F_6 + \cdots + F_{2n} + F_{2n+3}$

n winning moves, out of $F_{2n+3} = \lfloor \frac{\phi^{2n+3}}{\sqrt{5}} + 0.5 \rfloor$.
For $n = 10$, probability already 0.04%

What to do: Always pick 1, opponent has only 50-50 chance each time.