The Rényi-Ulam Pathological Liar Game with a Fixed Number of Lies

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Abstract

The q-round Rényi-Ulam pathological liar game with k lies on the set $[n] := \{1, \ldots, n\}$ is a 2-player perfect information zero sum game. In each round Paul chooses a subset $A \subseteq [n]$ and Carole either assigns 1 lie to each element of A or to each element of $[n] \setminus A$. Paul wins if after q rounds there is at least one element with k or fewer lies. The game is dual to the original Rényi-Ulam liar game for which the winning condition is that at most one element has k or fewer lies. Define $F_k^*(q)$ to be the minimum n such that Paul can win the q-round pathological liar game with k lies and initial set $[n]$. For fixed k we prove that $F_k^*(q)$ is within an absolute constant (depending only on k) of the sphere bound, $2^q/\binom{q}{\le k}$; this is already known to hold for the original Rényi-Ulam liar game due to a result of J. Spencer.

Key words: Rényi-Ulam game, pathological liar game, sphere bound, searching with lies
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1 Introduction

In this paper we consider the following 2-player perfect information zero-sum game, which we call the *Rényi-Ulam pathological liar game*, first defined in

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[4]. The players Paul and Carole play a q-round game on a set of n elements, $[n] := \{1, \ldots, n\}$. Each round, Paul splits the set of elements by choosing a *question* set $A \subseteq [n]$; Carole then completes the round by choosing to assign one *lie* either to each of the elements of A, or to each of the elements of $[n] \setminus A$. A given element is removed from play, or *disqualified*, if it accumulates $k + 1$ lies, where k is a predetermined nonnegative constant; in choosing the question set A, we may consider the game to be restricted to the *surviving* elements, which have $\leq k$ lies. The game starts with each element having no associated lies. If after q rounds at least one element survives, Paul wins; otherwise Carole wins. Thus Paul plays a strategy to preserve at least one element for q rounds, and Carole answers adversely. We think of a capricious or contrary Carole lying "pathologically" in order to disqualify elements as quickly as possible. Our main result, stated as Theorem 1 in Section 2 and proved in Section 4, is a tight asymptotic characterization of the minimum n for which Paul has a winning strategy for the q-round game with a fixed number, k, of lies.

This game arises as the dual to the Rényi-Ulam liar game, originating in [10] and [12], which we refer to as the *original liar game*. The simplest version of the original game is the "20 questions" game in which Paul may ask 20 Yes-No questions in order to identify a distinguished element x from a set $[n]$, where Carole answers "Yes" or "No" without lying. Here, Paul has a winning strategy iff $\log_2 n \leq 20$. In the general version, the number of rounds q and number of elements n are predetermined, as is the number, k, of times Carole is allowed to lie. We take the equivalent viewpoint that the distinguished element is not chosen ahead of time by Carole, but rather that she must answer consistently with there being at least one candidate for the distinguished element at each round. Thus a candidate element $y \in [n]$ cannot be the distinguished element if it would cause Carole to have lied about it $k + 1$ times. Paul's strategy in the original game, therefore, is to win by forcing Carole to associate $k+1$ lies with all but one element within q rounds, and Carole's strategy is to answer questions adversely so that at least two candidate elements remain after q rounds. Recently, Pelc thoroughly surveyed what is known about the original liar game and many of its variants [8].

In the pathological liar game at least one element must survive for Paul to win, but in the original game at most one element may survive for him to win. The remaining mechanics of the two games are the same, in that each round Paul chooses a question subset $A \subseteq [n]$ and Carole decides to assign lies either to A or to $[n] \setminus A$. The duality between the two games is due to the following fact which we clarify and prove in [5]: the original liar game is a relaxation of error-correcting codes (cf. [9]), and the pathological liar game is the corresponding relaxation of covering codes (cf. [2]), the well-known dual to error-correcting codes.

In Section 2, we describe how each stage of the pathological game can be encoded in a $(k+1)$-tuple state vector which keeps track of the number of lies associated with each element. In Section 3 we discuss the Berlekamp weight function on a state vector and how a winning strategy by Paul corresponds to maximizing (minimizing) the weight of the state vector after q rounds in the pathological (original) liar game. In Section 4, we prove that Paul can win if the initial state vector has sufficient weight; this yields the minimum value of n, up to a constant independent of q, for which Paul can win the q-round game with a fixed number, k, of lies. Section 5 concludes the paper with remarks on computing this minimum n exactly for specific values of k.

2 The vector game format

The mechanics of both the pathological liar game and the original liar game are encapsulated in the following vector framework due to Berlekamp [1]. Given that the game parameters are n elements, q rounds, and k lies, the initial state of the game is the $(k+1)$-vector $(n, 0, \ldots, 0)$. An intermediate stage of the game after some number of rounds is encoded by the *state vector* $x = (x_0, x_1, \ldots, x_k)$, where x_i denotes the number of elements of $[n]$ associated with i lies (disqualified elements, with $k+1$ lies, are not tracked by the state vector). The state vector completely encodes a stage of the game because an element of $[n]$ is distinguished only by the number of lies associated with it. Paul chooses a question set $A \subseteq [n]$ corresponding to an integer *question vector* $a = (a_0, a_1, \ldots, a_k)$ which must be *legal*, that is, $0 \le a_i \le x_i$ for each $i \in \{0, \ldots, k\}$. Carole answers either "Yes" or "No." By answering "Yes," Carole assigns an additional lie to each element in $[n] \setminus A$, so that the next state vector $Y(x, a)$ is obtained from x by moving elements corresponding to $[n] \setminus A$ to the right one position. Analogously, by answering "No," Carole causes the next state vector $N(x, a)$ to arise from moving elements corresponding to A to the right one position. Therefore the subsequent state chosen by Carole is either

$$\begin{aligned} Y(x,a) &:= (\quad a_0,\ a_1 + x_0 - a_0,\ \ldots,\ a_k + x_{k-1} - a_{k-1}) \quad \text{or} \\ N(x,a) &:= (\ x_0 - a_0,\ x_1 - a_1 + a_0,\ \ldots,\ x_k - a_k + a_{k-1}). \end{aligned} \tag{1}$$

Elements which become associated with $k+1$ lies are considered to be shifted out of the state vector to the right, and so we may consider the question set A and the set of elements $[n]$ to be restricted at any given stage to the surviving elements. In the pathological liar game, Paul wins iff after q rounds $\sum_{i=0}^{k} x_i \ge 1$ (at least one element survives). In the original liar game, Paul wins iff after q rounds $\sum_{i=0}^{k} x_i \le 1$.

More generally, we may consider a game starting with an arbitrary nonnegative state vector $x = (x_0, \ldots, x_k)$. We will use the following shorthand.

Definition 1 *(i) The* $(x,q,k)^*$-game *is the* q*-round pathological liar game with* k *lies and initial state* x*.*
(ii) The (x,q,k)-game *is the* q*-round original liar game with* k *lies and initial state* x*.*
In either game, the initial state $x=(x_0,\ldots,x_k)$ *encodes for* $0\le i\le k$ *the number* x_i *of elements which are initially associated with* i *lies.*

The k is redundant when x is specified. Both games are monotonic in the following sense. Suppose $x=(x_0,\ldots,x_k)$, $y=(y_0,\ldots,y_k)$, and $0\le y_i\le x_i$ for all $0\le i\le k$; i.e., x *covers* y. If Paul has a strategy to win the $(y,q,k)^*$-game (the (x,q,k)-game), then he has a strategy to win the $(x,q,k)^*$-game (the (y,q,k)-game). The new strategy is obtained from the winning strategy in the pathological game by arbitrarily choosing whether the extra elements corresponding to x_i-y_i are in A or $[n]\setminus A$, and in the original game by restricting all questions A by intersection with the set of all elements represented by $y_0,\ldots,y_k$. In fact, the same monotonicity holds if x *majorizes* y; i.e., if for all $0\le j\le k$, $\sum_{i=0}^{j} y_i\le\sum_{i=0}^{j} x_i$. This is because an element lasts longer in the game if it starts with fewer associated lies. We may now define $F_k^*(q)$ to be the minimum number n such that Paul has a winning strategy for the $((n,0,\ldots,0),q,k)^*$-game. The previously defined maximum n such that Paul can win the $((n,0,\ldots,0),q,k)$-game is $F_k(q)$. Pelc determined $F_1(q)$ exactly in [7], Guzicki determined $F_2(q)$ in [6], and Deppe determined $F_3(q)$ in [3]. Implicitly in Section 3 of [11], Spencer determined $F_k(q)$ for fixed k to within a constant independent of q. Since this result is of particular importance to this paper, we restate it now.

Theorem S1 (Spencer) *For any fixed nonnegative integer* k *there exist constants* q_k, C_k *such that for all* $q\ge q_k$*,*

$$\frac{2^q}{\binom{q}{\le k}}-C_k\ \le\ F_k(q)\ \le\ \frac{2^q}{\binom{q}{\le k}}.$$

An interpretation of the denominator $\binom{q}{\le k}:=\sum_{i=0}^{k}\binom{q}{i}$ will be given in the next section. The main result of this paper, which we prove in Section 4, is the following dual of Theorem S1.

Theorem 1 *For any fixed nonnegative integer* k *there exist constants* q_k^*, C_k^* *such that for all* $q\ge q_k^*$*,*

$$\frac{2^q}{\binom{q}{\le k}}\ \le\ F_k^*(q)\ \le\ \frac{2^q}{\binom{q}{\le k}}+C_k^*.$$

3 The Berlekamp weight function

For a nonnegative integer q and a state vector $x = (x_0, \ldots, x_k)$, the *q-weight* of x is defined to be

$$\mathrm{wt}_q(x) := \sum_{i=0}^{k} x_i \binom{q}{\le k-i}. \tag{2}$$

This is the *Berlekamp weight function* introduced in [1]. The number of ways to select positions for at most $k-i$ lies in a sequence of Y/N responses by Carole of length q is $\binom{q}{\le k-i}$, which motivates the weight of an element counted by x_i. We will see that Carole can always win the $(x, q, k)^*$-game when $\mathrm{wt}_q(x) < 2^q$. Intuitively, elements with fewer associated lies are worth more toward a win by Paul. To borrow an analogy from [11], we can think of the x_i's as representing *coins* of various denominations, where we call the coins with smallest weight, counted by x_k, *pennies*. We now present a well-known conservation lemma concerning the weight function, previously appearing in [1].

Lemma 2 (Conservation of weight) *Let $q \ge 1$. With state vector x and question vector a, legal for x, we have*

$$\mathrm{wt}_q(x) \;=\; \mathrm{wt}_{q-1}(Y(x,a)) + \mathrm{wt}_{q-1}(N(x,a)).$$

PROOF. Using (1) and (2), we compute

$$\mathrm{wt}_{q-1}(Y(x,a)) + \mathrm{wt}_{q-1}(N(x,a)) \;=\; x_0\binom{q-1}{\le k} + \sum_{i=1}^{k}(x_i + x_{i-1})\binom{q-1}{\le k-i}$$

$$= \sum_{i=0}^{k} x_i \left(\binom{q-1}{\le k-i} + \binom{q-1}{\le k-i-1}\right) \;=\; \mathrm{wt}_q(x),$$

by repeated use of the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\square$

In each round, Carole might always choose the resulting state with lower weight, giving a constraint on Paul's ability to win the $(x, q, k)^*$-game which holds for any k. We call the following lemma the *sphere bound* because of a connection to the sphere bound of coding theory to be made explicit in [5].

Lemma 3 (Sphere bound) *Let $q, k \ge 0$ and let $x = (x_0, \ldots, x_k)$ be a non-negative vector. If $\mathrm{wt}_q(x) < 2^q$, then Carole can win the q-round pathological liar game with k lies and initial state x. Consequently, $F_k^*(q) \ge 2^q / \binom{q}{\le k}$.*

PROOF. Regardless of Paul's initial question, by Lemma 2 Carole may respond so that the resulting state has weight at most $\mathrm{wt}_q(x)/2 < 2^{q-1}$. By induction, Carole may respond to Paul's remaining $q-1$ questions to ensure the 0-weight of the final state is < 1. Since the state vector must always be integer, Carole can always force the vector $(0, \ldots, 0)$ in q rounds. $\Box$

In the original game, the analog to the above lemma is that Carole has a strategy to win the (x, q, k)-game when $\mathrm{wt}_q(x) > 2^q$. This is proved in [11] by showing that if Carole answers randomly at each stage, the probability that the final weight is > 1 is nonzero, and thus Carole has a winning strategy since it is a perfect information game. The proof of Lemma 3 could be rewritten from this randomized perspective.

Lemma 3 shows that a necessary condition for Paul to win the q-round pathological liar game with starting state x is that $\mathrm{wt}_q(x) \geq 2^q$, but in general this is not sufficient. Paul is not always able to choose a question which balances the weights of the possible next states. Given some intermediate state x with $j+1$ rounds remaining and a question a, the resulting *weight imbalance* between possible next states is defined as (cf. Section 2 of [11])

$$\Delta_j(x, a) \ := \ \mathrm{wt}_j(Y(x, a)) - \mathrm{wt}_j(N(x, a)). \tag{3}$$

The following is a counterexample to the converse of Lemma 3.

Example 1 *Let* $x = (3, 1)$ *be the initial state of a* $((3, 1), 4, 1)^*$*-game. Note that* $\mathrm{wt}_4((3, 1)) = 3 \cdot 5 + 1 \cdot 1 = 16$*, and so Paul could possibly have a winning strategy. But any first-round question* $a = (a_0, a_1)$ *by Paul will satisfy* $|\Delta_3(x, a)| \geq 2$*. One question minimizing* $|\Delta_3(x, a)|$ *is* $a = (1, 1)$*, for which* $Y(x, a) = (1, 3)$*,* $N(x, a) = (2, 1)$*, and* $\Delta_3(x, a) = 7 - 9 = -2$*. In any event, Carole responds so that the next state has 3-weight at most 7, guaranteeing herself to win the game.*

Paul's goal in the pathological liar game, in terms of the weight function, corresponds to maximizing the 0-weight of the game state after q rounds. The capability to identify situations in which he can choose "perfectly balancing" questions at every stage so that $\Delta_j(x, a) = 0$ would provide a partial converse to Lemma 3; however, this is sometimes impossible (cf. Example 1), and difficult to know if it is possible when initially the q-weight is close to 2^q.

4 Asymptotics of the k-lie game

Since the full converse to Lemma 3 is impossible, we instead wish to identify the states x having $\mathrm{wt}_q(x)$ close to 2^q for which Paul can win the $(x, q, k)^*$-game. As Spencer proved in [11], there is a large set of states $x = (x_0, \ldots, x_k)$ such that if $\mathrm{wt}_q(x) = 2^q$ and the number of "pennies" x_k is large enough, then

Paul can find q questions which make the weight imbalance vanish *at each stage*. The number of pennies in the next state is maintained sufficiently by drawing from x_{k-1} and x_k. To employ Spencer's result, it will suffice to begin with x having q-weight slightly more than 2^q and reduce in k rounds to a state y with $(q-k)$-weight exactly 2^{q-k} for which Spencer's theorem holds. Here now is Spencer's result, essentially appearing as the "Main Theorem" in Section 2 of [11], in a form convenient for our purposes.

Theorem S2 (Spencer) *Let k be fixed. There are constants c, q_0 (dependent on k) so that the following holds for all $q \geq q_0$ and all nonnegative integer $x = (x_0, \dots, x_k)$: if $\mathrm{wt}_q(x_0, \dots, x_k) = 2^q$ and $x_k > cq^k$, then Paul has a strategy to reach a state z with $\mathrm{wt}_0(z) = 1$ in exactly q rounds such that every intermediate state $(u_0, \dots, u_k)$ after playing j rounds satisfies $\mathrm{wt}_{q-j}(u_0, \dots, u_k) = 2^{q-j}$.*

Theorem 4 *Let k be fixed. There are constants c_1, q_k^* (dependent on k) so that the following holds for all $q \geq q_k^*$ and all nonnegative integer $x = (x_0, \dots, x_k)$: if $\mathrm{wt}_q(x_0, \dots, x_k) \geq 2^q + c_1\binom{q}{k}$, then Paul can win the q-round pathological liar game with k lies and initial state $x = (x_0, \dots, x_k)$.*

PROOF. The proof proceeds in three main stages. First, the first k rounds of the game are played with a "floor-ceiling" question strategy which ensures that the resulting state y' satisfies $\mathrm{wt}_{q-k}(y') \geq 2^{q-k}$. Second, coins are removed from y' to obtain y with $(q-k)$-weight exactly 2^{q-k}. Finally, Theorem S2 is applied to y to reach a state z with $\mathrm{wt}_0(z) = 1$ after an additional $q-k$ rounds.

Paul plays the first k rounds of the game, reaching the state $y' = (y'_0, \dots, y'_k)$, according to the following strategy which is oblivious to Carole's responses. If $u(j) = (u_0(j), \dots, u_k(j))$ is the state when j rounds remain, then for $q \geq j > q-k$, Paul's next question $a(j) = (a_0(j), \dots, a_k(j))$ is defined by letting $a_i(j) = \lfloor u_i(j)/2 \rfloor$ or $\lceil u_i(j)/2 \rceil$, so that the overall choice of floors and ceilings for the odd $u_i(j)$'s alternates.

By combining (1) and (2) with the definition of Δ_j in (3), the weight imbalance of the two possible next states when $j+1$ rounds remain is at most

$$\Delta_j(u(j+1), a(j+1)) = \sum_{i=0}^{k}(2a_i(j+1) - u_i(j+1))\binom{j}{k-i} \leq \left|\binom{j}{k}\right|. \quad (4)$$

By Lemma 2 and (4), we have for each intermediate state $u(j+1)$ (with indexes $j+1$ suppressed for clarity)

$$\min\{\mathrm{wt}_j(\mathrm{Y}(u,a)), \mathrm{wt}_j(\mathrm{N}(u,a))\} \geq \frac{\mathrm{wt}_{j+1}(u) - \binom{j}{k}}{2}.$$

Therefore with an initial state of weight

$$\mathrm{wt}_q(x) \geq 2^q + c_1\binom{q}{k} \geq 2^q + \sum_{j=q-1}^{q-k} 2^{q-1-j}\binom{j}{k},$$

for some constant c_1 and $q \geq q_1$ large enough, Paul can guarantee a state y' with $\mathrm{wt}_{q-k}(y') \geq 2^{q-k}$ after k rounds.

The number of pennies y'_k after k rounds is large, by the following argument. Since $\mathrm{wt}_q(x) \geq 2^q$ and the largest weight of an element is $\binom{q}{\leq k} \leq q^k$, then $\sum_{i=0}^{k} x_i \geq 2^q/q^k$. Thus there exists a coordinate i_0 for which $x_{i_0} \geq 2^q/\big((k+1)q^k\big)$. By definition of the first k questions,

$$\begin{aligned} y'_k = u_k(q-k) &\geq \lfloor 2^{-1}u_k(q-k+1)\rfloor \geq \cdots \geq \lfloor 2^{-i_0}u_k(q-k+i_0)\rfloor \\ &\geq \lfloor 2^{-i_0-1}u_{k-1}(q-k+i_0+1)\rfloor \geq \cdots \geq \lfloor 2^{-k}u_{i_0}(q)\rfloor = \lfloor 2^{-k}x_{i_0}\rfloor \\ &\geq \left\lfloor 2^{-k}\cdot\frac{2^q}{(k+1)q^k}\right\rfloor \geq c_2 q^k. \end{aligned}$$

The first line is true because $u_k(j)$ is at least $\lfloor u_k(j+1)/2\rfloor$, the second line is true because $u_i(j)$ is at least $\lfloor u_{i-1}(j+1)/2\rfloor$, and the last inequality is true for any choice of c_2 and $q \geq q_2$ provided q_2 is taken to be large enough. We note that the choice of c_1 does not affect the choice of c_2 in this analysis.

Now obtain the state $y = (y_0, \ldots, y_k)$ with $(q-k)$-weight 2^{q-k} from y' by greedily removing coins of decreasing weight, so that either only 2^{q-k} pennies are left, or fewer than $\binom{q-k}{\leq k}$ pennies were removed. In the first case Paul trivially can make the game last another $q-k$ rounds; in the second case at least

$$y_k \geq c_2 q^k - \binom{q-k}{\leq k} \geq c_3(q-k)^k$$

pennies remain. The constant c_3 can be chosen to be at least c_2-1, for instance, provided that $q \geq q_3$ for q_3 large enough. Choose c_3 and $q_k^* \geq \max\{q_1, q_2, q_3\}$ large enough so that c_3 and $q_k^* - k$ satisfy the requirements of Theorem S2 for the $(y, q-k, k)$-game. Therefore Paul can win the $(x, q, k)^*$-game. □

PROOF. (Proof of Theorem 1.) From Lemma 3, $F_q^*(k) \geq 2^q/\binom{q}{\leq k}$. Now suppose $q \geq q_k^*$ and let $n = \lceil (2^q + c_1\binom{q}{k})/\binom{q}{\leq k}\rceil$, where c_1 and q_k^* are as in Theorem 4. Then $\mathrm{wt}_q(n, 0, \ldots, 0) \geq 2^q + c_1\binom{q}{k}$ and $F_k^*(q) \leq n \leq \lceil (2^q + c_1\binom{q}{k})/\binom{q}{\leq k}\rceil \leq 2^q/\binom{q}{\leq k} + C_k^*$ for $q \geq q_k^*$ and some constant C_k^*. □

5 Concluding remarks

The exact excess weight above 2^q required in Theorem 4 is difficult to compute for general k. However, for small k, $F_k^*(q)$ may be determined by following the framework of the three stages of the proof of Theorem 4, carefully tracking the first k rounds. For $k=1$, the only issue is whether the n in the initial state $(n,0)$ is even or odd, since n odd forces a nonzero weight imbalance in the first round. After the first round, Paul has enough pennies to make the weight imbalance vanish at each subsequent stage, which leads to the following.

Theorem 5 *Let $q \geq 0$. Paul has a winning strategy for the q-round pathological liar game with 1 lie and initial state $(n,0)$ iff*

$$2^q \leq \begin{cases} n(q+1) & \text{if } n \text{ is even,} \\ n(q+1)-(q-1) & \text{if } n \text{ is odd.} \end{cases}$$

A formula for $F_1^*(q)$ is immediate. Full details of the proof of Theorem 5, and of the more complex statement and proof for $k=2$, are available in [5]

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