Asymptotic Formula for $(1 + 1/x)^x$, Revisited

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In a recent note [**1**] appearing in this *Monthly*, Chen and Choi calculated a_j in

$$\left(1 + \frac{1}{x}\right)^x = e \sum_{j=0}^{\infty} \frac{a_j}{x^j} \tag{1}$$

We offer a briefer calculation of these same coefficients. Applying the change of variables $y = \frac{1}{x}$, and the two functions $f(y) = e^y$, $g(y) = \frac{\ln(1+y)}{y}$, we rewrite (1) as

$$f(g(y)) = \sum_{j=0}^{\infty} (ea_j) y^j$$

We take derivatives of this formal power series j times, and substitute $y = 0$, to get

$$\frac{d^j}{dy^j} f(g(y))|_{y=0} = e(j!)a_j \tag{2}$$

The left side of (2) may be calculated with Faà di Bruno's famous formula, which gives

$$\frac{d^j}{dy^j} f(g(y)) = \sum \frac{j!}{k_1! \cdots k_j!} f^{(k_1+\cdots+k_j)}(g(y)) \left(\frac{g^{(1)}(y)}{1!}\right)^{k_1} \cdots \left(\frac{g^{(j)}(y)}{j!}\right)^{k_j}$$

where the sum is taken over all solutions to $k_1 + 2k_2 + \cdots + jk_j = j$. Because $g(y) = 1 - \frac{y}{2} + \frac{y^2}{3} - \frac{y^3}{4} + \cdots$, we have $g^{(t)}(y)|_{y=0} = (-1)^t \frac{t!}{t+1}$. We also have $\lim_{y \to 0} f^{(k_1+\cdots+k_j)}(g(y)) = \lim_{y \to 0} f(g(y)) = e$. Combining, we get

$$e(j!) \sum \frac{1}{k_1! \cdots k_j!} \left(\frac{1}{2}\right)^{k_1} \cdots \left(\frac{1}{j+1}\right)^{k_j} (-1)^{k_1+2k_2+\cdots+jk_j} = e(j!)a_j$$

We now cancel $e(j!)$ from both sides, and use the $k_1 + 2k_2 + \cdots + jk_j = j$ restriction, to get the main result from [**1**]:

$$(-1)^j \sum \frac{1}{k_1! \cdots k_j!} \left(\frac{1}{2}\right)^{k_1} \cdots \left(\frac{1}{j+1}\right)^{k_j} = a_j$$

REFERENCES

1. C.-P. Chen and J. Choi, An Asymptotic Formula for $(1 + 1/x)^x$ Based on the Partition Function, *Amer. Math. Monthly* **121** (2014) 338-343.