Accepted Elasticity in Local Arithmetic Congruence Monoids

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http://www-rohan.sdsu.edu/~vadim/accepted-talk.pdf

Shameless advertising

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This work was done in Summer 2012, jointly with undergraduates Lorin Crawford, Jason Steinberg, and Marla Williams.

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Arithmetic Congruence Monoids

Fix $a, b \in \mathbb{N}$ with $a^2 \equiv a \pmod{b}$.

Set $S = \{x \in \mathbb{N} : x \equiv a \pmod{b}\} \cup \{1\}$.

S is a multiplicative submonoid of $\mathbb{N}$ called an ACM.

Famous example: $a = 1, b = 4$, "Hilbert monoid"

Several recent papers have studied ACM arithmetic.

This work considers one property, in the one class not yet understood, called "local".

$\gcd(a, b) = p^\alpha$

i.e. $a = p^\alpha \xi, b = p^\alpha n$, with $\gcd(\xi, n) = \gcd(p, n) = 1$.

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Accepted Elasticity

For ACM S, and $x \in S \setminus \{1\}$, we may write $x = x_1 \cdots x_k$, where x_k are irreducible.

We call k the *length* of this factorization into irreducibles.

We call the *elasticity* of x the ratio of the maximum possible length to the minimum possible length.

We say S has *accepted elasticity* if there is some $x \in S$ whose elasticity is maximal, over all elements of S.

Does a given ACM have accepted elasticity?

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General Approach

Recall $S = \{x \in \mathbb{N} : x \equiv p^\alpha \xi \pmod{p^\alpha n}\} \cup \{1\}$.

It is very useful to consider the group of units $(\mathbb{Z}/n\mathbb{Z})^\times$.

What are p, p^α congruent to? How big is α?
What is α congruent to, modulo $\phi(n)$?

Note: $p^\alpha \xi \equiv 1 \pmod{n}$, so ξ is vestigial.

Main Theorem

Recall $S = \{x \in \mathbb{N} : x \equiv p^\alpha \xi \pmod{p^\alpha n}\} \cup \{1\}$.

This has accepted elasticity for all p and for all sufficiently large α if:

1. $n \in \{1, 2, 8, 12\}$, or
2. One of $\{qrs, 4qr, 8q\}$ divides n, or
3. $n \in \{q^s r^t, 2q^s r^t\}$ with $\gcd(q-1, r-1) > 2$.

   $(q, r, s$ odd primes)

Otherwise, ∞ many p have accepted elasticity for all suff. large α, and ∞ many p do not, for ∞ many α.

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For Further Reading

P. Baginski, S. Chapman
Arithmetic Congruence Monoids: A Survey (under review)

L. Crawford, VP, J. Steinberg, M. Wlliams
Accepted Elasticity in Local ACMs (under review)

M. Jenssen, D. Montealegre, VP
Irreducible Factorization Lengths and the Elasticity Problem Within $\mathbb{N}$, *American Math Monthly* 120 (4) 2013, pp. 322-328.

A. Fujiwara, J. Gibson, M. Jenssen, D. Montealegre, VP, Ari Tenzer
Arithmetic of Congruence Monoids (under review)

C. Allen, VP, W. Radil, R. Rankin, H. Williams
Full Elasticity in Local ACMs (in preparation)

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