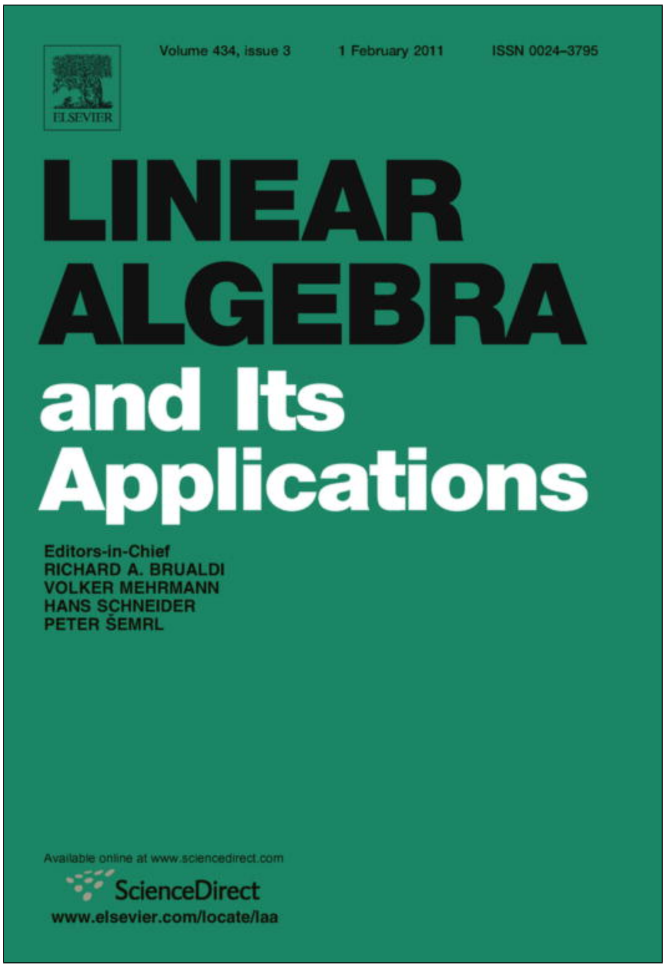
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Number theory of matrix semigroups

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Unlike factorization theory of commutative semigroups which are well-studied, very little literature exists describing factorization properties in noncommutative semigroups. Perhaps the most ubiquitous noncommutative semigroups are semigroups of square matrices and this article investigates the factorization properties within certain subsemigroups of $M_n(\mathbb{Z})$, the semigroup of $n \times n$ matrices with integer entries. Certain important invariants are calculated to give a sense of how unique or non-unique factorization is in each of these semigroups.

1. Introduction

Many number theoretic results rely on knowing whether or not certain algebraic structures are endowed with the property of unique factorization. This paper investigates questions of unique factorization within certain classes of integral-valued matrices. Given a class S of matrices, we call $M \in S$

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irreducible if it cannot be written as $M = AB$ where $A, B \in S$ are noninvertible $n \times n$ matrices. We intend to answer the following questions:

1. Can we enumerate the set of irreducibles of S?
2. Given a matrix $A \in S$, what can we say about the factorizations of A into irreducibles?

Such questions of matrix factorizations were studied by Cohn [4] as early as 1963. In the context of semigroups of matrices, factorization problems were studied in 1966 by Jacobson and Wisner [14] and in 1985 by Ch'uan and Chuan [2]. Motivated by these results, we apply the concepts of contemporary factorization theory to semigroups of integral-valued matrices. The study of non-unique factorizations has been well developed over the past several decades and was unified in [7]. We intend to use methods from commutative contexts to study factorizations of matrices with integer entries.

Throughout, $\mathbb{N}$ will denote the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A semigroup is a pairing $(S, \cdot)$ where S is a set and $\cdot$ is an associative binary operation on S. When the binary operation is clear from context and $A, B \in S$, we will simply write AB instead of $A \cdot B$. If S contains an element I such that $AI = IA = A$ for all $A \in S$, then I is the *identity* of S. An element $A \in S$, a semigroup with identity, is a unit of S if there exists an element $B \in S$ such that $AB = BA = I$. A nonunit $A \in S$ is called an *atom* of S if whenever $A = BC$ for some elements $B, C \in S$, either B or C is a unit of S. The semigroup S is said to be *atomic* provided each nonunit element in S can be written as a product of atoms of S.

We now briefly introduce some important invariants that we will use throughout to describe how unique factorizations are within a given semigroup. Let S denote an atomic semigroup and let A be a nonunit element of S. The set

$$\mathcal{L}(A) = \{t : A = A_1A_2 \cdots A_t \text{ with each } A_i \text{ an atom of } S\}$$

is the *set of lengths of* A. We denote by $L(A) = \sup \mathcal{L}(A)$ the longest (if finite) factorization length of A and $l(A) = \min \mathcal{L}(A)$ the minimum factorization length of A. The *elasticity*, denoted $\rho(A) = \frac{L(A)}{l(A)}$, of A gives a coarse measure of how far away A is from having unique factorization. Indeed, if A has a unique factorization $A = A_1A_2 \cdots A_t$, then $\mathcal{L}(A) = \{t\}$ and hence $l(A) = L(A) = t$ and $\rho(A) = t/t = 1$. The *elasticity* of the semigroup S, $\rho(S)$ is given by $\rho(S) = \sup\{\rho(A) : A \in S\}$. If $\mathcal{L}(A) = \{t_1, t_2, \ldots\}$ with $t_i < t_{i+1}$ for each i, then $\Delta(A) = \{t_{i+1} - t_i : t_i, t_{i+1} \in \mathcal{L}(A)\}$. Then $\Delta(S) = \bigcup_{A \in S} \Delta(A)$.

If S is an atomic semigroup and T is an atomic semigroup with identity, a surjective semigroup homomorphism $\phi : S \to T$ is a *transfer homomorphism* provided that whenever $\phi(s) = xy$ with x and y nonunits of T, there exist a and b nonunits of S such that $\phi(a) = ux$, $\phi(b) = vy$ with u, v units of S and $ab = s$. It is well known (cf. [7]) that if $\phi : S \to T$ is a transfer homomorphism between two atomic semigroups, then $\rho(S) = \rho(T)$ and $\Delta(S) = \Delta(T)$. Transfer homomorphisms will allow us, in Sections 2 and 5, to understand the factorization of certain matrix semigroups by studying factorizations of elements in certain simpler semigroups.

We say that an atomic semigroup is *factorial* if every factorization is unique up to units. Note that this differs slightly from the commutative definition since we consider $AB = BA$ to be two distinct factorizations of the same element. An atomic semigroup is *half-factorial* provided $L(A) = l(A)$ for each nonunit $A \in S$. Finally, following [1], we define an atomic semigroup to be *bifurcus* provided $l(A) = 2$ for every nonunit non-atom A of S. We note that if S is bifurcus, then $\rho(S) = \infty$ and $\Delta(S) = \{1\}$.

Finally, we introduce some basic notation pertaining to greatest common divisor of a matrix. If A is any $m \times n$ matrix, we set $\gcd(A)$ to be the greatest common divisor of the mn entries of A.

The authors would like to thank the referee for a careful reading of this paper and for suggesting several generalizations which now occur in Sections 2 and 5.

2. The semigroups $M_n(\mathbb{Z})$ and $T_n(\mathbb{Z})$

Before considering specific subsemigroups of matrices, we use this section to consider the semigroup $M_n(\mathbb{Z})$ of all $n \times n$ integer valued matrices and the subsemigroup $T_n(\mathbb{Z})$ of $n \times n$ upper triangular integer valued matrices. Such matrices are widely studied in mathematics; see, for example [12,18]. In this section we will find that the factorization of a matrix with integer entries is equivalent to the factorization of its determinant in $\mathbb{Z}$. Thus we are able to relate factorization of matrix semigroups with the better-understood integer semigroups as studied in [10]. First we give a preliminary result which we use to classify the units in $M_n(\mathbb{Z})$ and $T_n(\mathbb{Z})$.

Lemma 2.1. *Let S denote a subsemigroup of $M_n(\mathbb{Z})$ containing I_n and let $A \in S$. If A is a unit of S, then $|\det(A)| = 1$. Moreover, if S is closed under adjugates, then $|\det(A)| = 1$ implies that A is a unit of S.*

Proof. If A is a unit, then $AB = I_n$ for some $B \in S$. Since $\det(A)\det(B) = \det(I_n) = 1$, and since all of the entries of A are integers, $\det(A) = \pm 1$.

Suppose now that $\det(A) = \pm 1$ and that $\text{adj}(A) \in S$. Then A is invertible in $M_n(\mathbb{R})$ and $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) \in S$. □

In particular, since $M_n(\mathbb{Z})$ and $T_n(\mathbb{Z})$ are closed under adjugates, the units of these semigroups are precisely the elements with determinant plus or minus one.

The following lemma provides transfer homomorphisms from $M_n(\mathbb{Z})$ and $T_n(\mathbb{Z})$ to the semigroup $\mathbb{Z}$, thus allowing us to study the factorization properties of a matrix A by instead studying the factorization properties of $\det(A) \in \mathbb{Z}$.

Lemma 2.2

1. *Let $A \in M_n(\mathbb{Z})$ such that $\det(A) = xy$ for some integers x and y. Then there exist $X, Y \in M_n(\mathbb{Z})$ such that $\det(X) = x$, $\det(Y) = y$ and $XY = A$.*
2. *Let $A \in T_n(\mathbb{Z})$ such that $\det(A) = xy$ for some integers x and y. Then there exist $X, Y \in T_n(\mathbb{Z})$ such that $\det(X) = x$, $\det(Y) = y$ and $XY = A$.*

Proof. Let $A \in M_n(\mathbb{Z})$ and write A using the Smith Normal Form as $A = UDV$ where $D \in M_n(\mathbb{Z})$ is a diagonal matrix and both U and V are elements of $M_n(\mathbb{Z})$ with determinant in the set $\{-1, 1\}$. If necessary, we can multiply the first rows of D and V by -1 in order to guarantee that $\det(D) = xy$. We can then construct D_1 and D_2 diagonal matrices with $D = D_1D_2$, $\det(D_1) = x$ and $\det(D_2) = y$. Now $A = XY$ where $X = UD_1$ and $Y = D_2V$ have the desired properties, proving (1).

Now let $A \in T_n(\mathbb{Z})$ and let $a_1, a_2, \ldots, a_n$ denote the diagonal entries of A. Note that $\det(A) = xy = a_1a_2 \cdots a_n$ and let $y = p_1p_2 \cdots p_m$ denote the prime factorization of y with the p_i not necessarily distinct primes. We now use induction on m. If $m = 1$, then $y = p_1$, a prime. Let $k \in \{1, 2, \ldots, n\}$ be the smallest positive integer such that $p_1 | a_k$. We claim that A can be factored as

$$A = \begin{bmatrix} T_1 & \mathbf{u} & V \\ 0 & a_k & W \\ 0 & 0 & T_2 \end{bmatrix} = \begin{bmatrix} T_1 & \mathbf{u}_1 & V \\ 0 & \frac{a_k}{p_1} & W \\ 0 & 0 & T_2 \end{bmatrix} \begin{bmatrix} I_{k-1} & \mathbf{u}_2 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & I_{n-k} \end{bmatrix},$$

where $T_1 \in T_{k-1}(\mathbb{Z})$, $T_2 \in T_{n-k}(\mathbb{Z})$, $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^{k-1}$, $V \in M_{k-1,n-k}(\mathbb{Z})$, and $W \in M_{1,n-k}(\mathbb{Z})$. Such a decomposition of A exists if and only if we can find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $T_1\mathbf{u}_2 + p_1\mathbf{u}_1 = \mathbf{u}$. That is, there must exist $\mathbf{u}_1 = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,k-1} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} u_{2,1} \\ u_{2,2} \\ \vdots \\ u_{2,k-1} \end{bmatrix}$ such that

$$\begin{bmatrix} a_1 & a_{12} & \cdots & a_{1(k-1)} \\ 0 & a_2 & \cdots & a_{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k-1} \end{bmatrix} \begin{bmatrix} u_{2,1} \\ u_{2,2} \\ \vdots \\ u_{2,k-1} \end{bmatrix} + p_1 \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,k-1} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{k-1} \end{bmatrix}.$$

Since a_{k-1} and p_1 are relatively prime there must exist $u_{1,k-1}$ and $u_{2,k-1}$ such that $u_{k-1} = p_1u_{1,k-1} + a_{k-1}u_{2,k-1}$. Now since a_{k-2} and p_1 are relatively prime we can find $u_{2,k-2}$ and $u_{1,k-2}$ such that $u_{k-2} = p_1u_{1,k-2} + a_{k-2}u_{2,k-2} + a_{k-2,k-1}u_{2,k-1}$.

Assume now that the result holds whenever y is the product of less than m primes. Finally, suppose that $y = p_1 \cdots p_m$ with each p_i prime. By our induction hypothesis, we see that A factors as $A = \widehat{X}\widehat{Y}$ where $\det(\widehat{X}) = xp_1$ and $\det(\widehat{Y}) = p_2 \cdots p_m$. Working as in the case $y = p_1$ above, we see that $\widehat{X}$ factors as $X\widetilde{Y}$ where $\det(X) = x$ and $\det(\widetilde{Y}) = p_1$. Setting $Y = \widetilde{Y}\widehat{Y}$, we see that A factors as $A = XY$ with $\det(X) = x$ and $\det(Y) = p_1 p_2 \cdots p_m = y$. This proves (2). $\square$

The transfer homomorphism $A \mapsto \det(A)$ provided by Lemma 2.2 together with the uniqueness of factorization in $\mathbb{Z}$ and $\mathbb{N}$ provide the following theorem.

Theorem 2.3. *Let S be one of the following subsemigroups of $M_n(\mathbb{Z})$:*

- $M_n(\mathbb{Z})$,
- $T_n(\mathbb{Z})$,
- $\{A \in M_n(\mathbb{Z}) \mid \det(A) > 1\}$, *or*
- $\{A \in T_n(\mathbb{Z}) \mid \det(A) > 1\}$.

Then

1. *A is an atom of S if and only if $\det(A)$ is prime,*
2. *$L(A) = l(A)$ is the number of (not necessarily distinct) prime factors of $\det(A)$, and*
3. *S is half-factorial.*

The remaining sections will consider certain subsemigroups of $M_n(\mathbb{Z})$ and $T_n(\mathbb{Z})$. As we shall see, these subsemigroups have more interesting factorization properties than the larger semigroups in which they live.

3. Semigroups which admit a transfer homomorphism to $\mathbb{Z}$

In this section we consider subsemigroups of $M_n(\mathbb{Z})$ and $T_n(\mathbb{Z})$ with various restrictions on the determinant. In Section 3.1 we study matrices whose determinants are divisible by a fixed positive integer k. We refer to these semigroups as ${}_kM_n(\mathbb{Z})$ and ${}_kT_n(\mathbb{Z})$, respectively. Such matrices have been considered in [3,5]. In Section 3.5 we study the larger class of matrices whose determinants are composite. Our main tools in studying factorization in these semigroups are the transfer homomorphisms provided by Lemma 2.2. In fact, these results hold in any semigroup which admits a transfer homomorphism to $\mathbb{Z}$ and thus we state all but the final corollaries in these sections in terms of this generality.

3.1. Semigroups which admit a transfer homomorphism to $k\mathbb{Z}$

Throughout this section, let $k > 1$ be a fixed positive integer and let S denote a semigroup which admits a transfer homormorphism $\phi : S \to \mathbb{Z}$ such that $\phi(S) = k\mathbb{Z}$. Note that since $1 \notin \phi(S)$, S contains no units. For an element $B \in S$, we set $\nu_r(B)$ to denote the number $\max\{t : r^t | \phi(B)\}$.

Lemma 3.1. *Let $k > 1$ be a positive integer and let S be a semigroup such that there exists a transfer homomorphism $\phi : S \to \mathbb{Z}$ such that $\phi(S) = k\mathbb{Z}$. If $A \in S$, then*

1. *$L(A) = \nu_k(A)$ and*
2. *A is an atom of S if and only if $k^2 \nmid \det(A)$.*

Proof. Suppose that A factors as $A = A_1 A_2 \cdots A_m$ in S. By the definition of S, $k | \phi(A_i)$ for all i and hence $k^m | \phi(A)$. Therefore, $L(A) \leqslant \nu_k(A)$. Using the transfer homomorphism, there must exist $A_1, A_2, \ldots, A_{\nu_k(A)} \in S$ with $\phi(A_i) = k$ for all $i \in \{1, 2, \ldots, \nu_k(A) - 1\}$ and $\phi(A_{\nu_k(A)}) = \frac{\phi(A)}{k^{\nu_k(A)-1}}$. Therefore, $L(A) \geqslant \nu_k(A)$. This proves (1), of which (2) is a consequence. $\square$

Lemma 3.2. *Let $k = p^t$ for some prime integer p and some positive integer t and let S be a semigroup such that there exists a transfer homomorphism $\phi : S \to \mathbb{Z}$ such that $\phi(S) = k\mathbb{Z}$. If $A \in S$, then $\mathsf{l}(A) = \left\lfloor \frac{\nu_p(A)+2t-2}{2t-1} \right\rfloor$.*

Proof. Write $\phi(A) = p^m x$ where $m \geqslant t$ and $p \nmid x$; that is, $\nu_p(A) = m$. Factor A in S as $A = A_1A_2 \cdots A_s$ where each A_i is an atom of S and set, for each i, $t_i = \nu_p(A_i)$. Since $\phi(A) = \phi(A_1)\phi(A_2)\cdots\phi(A_s)$, $m = \sum_{i=1}^{s} t_i$. Moreover, since each A_i is an atom, Lemma 3.1 gives us that $t_i \leqslant 2t-1$ for each $i \in \{1, 2, \ldots, s\}$. Thus $m = \sum_{i=1}^{s} t_i \leqslant s(2t-1)$. If $s \leqslant \left\lfloor \frac{\nu_p(A)+2t-2}{2t-1} \right\rfloor - 1$, then $s \leqslant \frac{\nu_p(A)-1}{2t-1}$ and hence $s(2t-1) \leqslant m-1$, a contradiction. Therefore, $l(A) \geqslant \left\lfloor \frac{\nu_p(A)+2t-2}{2t-1} \right\rfloor$.

Appealing to the transfer homomorphism, we see that factoring A is equivalent to factoring $\phi(A)$ in $\mathbb{Z}$. Write $\nu_p(A) = q(2t-1) + r$ where $0 \leqslant r < 2t-1$. Then $\left\lfloor \frac{\nu_p(A)+2t-2}{2t-1} \right\rfloor = q + 1 + \left\lfloor \frac{r-1}{2t-1} \right\rfloor$. Clearly, A is an atom of S if and only if $q = 0$. If A is not an atom then we can factor $\phi(A)$ as

$$\phi(A) = \begin{cases} (p^{2t-1})(p^t)(p^{r+t-1}x) & \text{if } r > 0, \\ (p^{2t-1})^{q-1}(p^{2t-1}x) & \text{if } r = 0 \end{cases}$$

giving either a factorization of length $q+1$ when $r > 0$ or a factorization of length q when $r = 0$. This proves that $\mathsf{l}(A) \leqslant \left\lfloor \frac{\nu_p(A)+2t-2}{2t-1} \right\rfloor$. □

Using Lemma 3.2, the following theorem gives the elasticity $\rho(S)$ regardless of k. To see this, it is important to recall that bifurcus semigroups necessarily have infinite elasticity.

Theorem 3.3. *Let $k > 1$ be a positive integer and let S be a semigroup such that there exists a transfer homomorphism $\phi : S \to \mathbb{Z}$ such that $\phi(S) = k\mathbb{Z}$:*

1. *If $k = p^t$ for some prime p and some positive integer t, then $\rho(S) = \frac{2t-1}{t}$.*
2. *If $k = st$ with $s, t > 1$ and $\gcd(s, t) = 1$, then S is bifurcus.*
3. *S is half-factorial if and only if k is prime.*

Proof. Suppose that $k = p^t$ for some prime p and some positive integer t. By Lemma 3.1, for any $A \in S$ we have that

$$\rho(A) = \frac{\left\lfloor \frac{\nu_p(A)}{t} \right\rfloor}{\left\lfloor \frac{\nu_p(A)+2t-2}{2t-1} \right\rfloor} \leqslant \frac{\frac{\nu_p(A)}{t}}{\frac{\nu_p(A)}{2t-1}} = \frac{2t-1}{t}.$$

Therefore, $\rho(S) \leqslant \frac{2t-1}{t}$. However, this elasticity is achieved by any element $A \in S$ with $\phi(A) = (p^t)^{2t-1} = (p^{2t-1})^t$ and hence $\rho(S) = \frac{2t-1}{t}$.

Suppose that $k = st$ with $s, t > 1$ and $\gcd(s, t) = 1$. If $A \in S$ can be written as the product of two elements in S, then by Lemma 3.1 $(st)^2 | \phi(A)$ and hence $\phi(A) = (st)^m x_1 x_2$ where $m \geqslant 2$, $s \nmid x_1$ and $t \nmid x_2$. Writing $\phi(A) = (s^{m-1}tx_2)(st^{m-1}x_1)$, we see that there exist matrices B and C in S such that $\phi(B) = s^{m-1}tx_2$, $\phi(C) = st^{m-1}x_1$ and $A = BC$. Since the images of B and C are not divisible by $(st)^2$, by Lemma 3.1 B and C are atoms of S and hence $l(A) = 2$.

If $k = st$ where $\gcd(s, t) = 1$ then S is bifurcus and hence S is not half-factorial. If $k = p^t$ for some prime p and some positive integer k, then $\rho(S) = \frac{2t-1}{t}$ and hence S is half-factorial if and only if $t = 1$; i.e., k is prime. □

The following corollary is an immediate result of applying the transfer homomorphism given in Lemma 2.2 to the results of this section.

Corollary 3.4. *Let $k > 1$ be a positive integer and either let $S = {}_kM_n(\mathbb{Z})$ or $S = {}_kT_n(\mathbb{Z})$:*

1. *If $k = p^t$ for some prime p and some positive integer t, then $\rho(S) = \frac{2t-1}{t}$.*
2. *If $k = st$ with $s, t > 1$ and $\gcd(s, t) = 1$, then S is bifurcus.*
3. *S is half-factorial if and only if k is prime.*

3.2. Composite images

In this section we consider semigroups S which admit a transfer homomorphism $\phi : S \to \mathbb{Z}$ such that every element in $\phi(S)$ is composite. We note that since $1 \notin \phi(S)$, S contains no units. For convenience, we let $r(A)$ denote the number of prime divisors of $\phi(A)$ counted with multiplicity and note that if $X, Y \in S$, then $r(XY) = r(X) + r(Y)$. Also note that if $A \in S$, then $r(A) \geqslant 2$. The following lemma is an analog of Lemmas 2.2 and 2.2 which will be allow us to compute both $\rho(S)$ and $\Delta(S)$.

Lemma 3.5. *Let S denote a semigroup which admits a transfer homomorphism $\phi : S \to \mathbb{Z}$ such that every element in $\phi(S)$ is composite:*

1. *A is an atom of S if and only if $r(A) \leqslant 3$.*
2. *If $r(A) = x + y$ with $x, y \geqslant 2$, then there exist $X, Y \in S$ with $r(X) = x$, $r(Y) = y$ and $A = XY$.*

Proof. Suppose that A factors as $A = BC$ in S. Since $B, C \in S$, we have that $r(B), r(C) \geqslant 2$. Then $r(A) = r(B) + r(C) \geqslant 2 + 2 = 4$.

Now suppose that $r(A) \geqslant 4$. Then we can write $r(A) = x + y$ with $x, y \geqslant 2$. We then factor $\phi(A) = p_1 \cdots p_x p_{x+1} \cdots p_{x+y}$ where the p_i are not necessarily distinct primes. Utilizing the transfer homomorphism, there exist elements X and Y in S with $\phi(X) = p_1 \cdots p_x$ and $\phi(Y) = p_{x+1} \cdots p_{x+y}$ and $A = XY$. Since $x, y \geqslant 2$, $\phi(X)$ and $\phi(Y)$ are composite and hence $X, Y \in S$. Moreover, A is not an atom of S. □

Theorem 3.6. *Let S denote a semigroup which admits a transfer homomorphism $\phi : S \to \mathbb{Z}$ such that every element in $\phi(S)$ is composite and let A be a non-atom of S. Then*

1. *$L(A) = \left\lfloor \frac{r(A)}{2} \right\rfloor$,*
2. *$l(A) = \left\lceil \frac{r(A)}{3} \right\rceil$,*
3. *$\rho(S) = \frac{3}{2}$, and*
4. *$\Delta(S) = 1$.*

Proof. Write $r(A) = 2\left\lfloor \frac{r(A)}{2} \right\rfloor + x$ where $x \in \{0, 1\}$. Suppose that $A = A_1A_2 \cdots A_s$ where each A_i is an atom of S. Then $r(A) = \sum_{i=1}^{s} r(A_i)$. As $r(A_i) \geqslant 2$ for each i, $r(A) \geqslant 2s$. Therefore, $L(A) \leqslant \left\lfloor \frac{r(A)}{2} \right\rfloor$. From Lemma 3.5 we know that since $r(A) = 2\left\lfloor \frac{r(A)}{2} \right\rfloor + x \geqslant 2$, we can factor A as $A_1A_2 \cdots A_{\left\lfloor \frac{r(A)}{2} \right\rfloor}$ where $r(A_i) = 2$ for all $i \in \left\{1, \ldots, \left\lfloor \frac{r(A)}{2} \right\rfloor - 1\right\}$ and $r\left(A_{\left\lfloor \frac{r(A)}{2} \right\rfloor}\right) = 2 + x$. Therefore, $L(A) = \left\lfloor \frac{r(A)}{2} \right\rfloor$.

Set $r(A) = 3k + x$ where $k \in \mathbb{N}_0$ and $x \in \{0, 1, 2\}$. If $x = 0$, we factor A as $A = A_1 \cdots A_k$ where $r(A_i) = 3$ for all i, $1 \leqslant i \leqslant k$. If $x \in \{1, 2\}$, we factor A as $A = A_1 \cdots A_{k+1}$ where $r(A_i) = 3$ for all i, $1 \leqslant i \leqslant k - 1$, $r(A_k) = 1 + x$ and $r(A_{k+1}) = 2$. This gives $l(A) \leqslant \lceil \frac{r(A)}{3} \rceil$. Next, suppose that $x \in \{1, 2\}$ and factor A as $A = A_1A_2 \cdots A_{\left\lfloor \frac{r(A)}{3} \right\rfloor} A_{\left\lfloor \frac{r(A)}{3} \right\rfloor + 1}$ where $r(A_i) = 3$ for all $i \in \left\{1, \ldots, A_{\left\lfloor \frac{r(A)}{3} \right\rfloor}\right\}$, $r\left(A_{\left\lfloor \frac{r(A)}{3} \right\rfloor}\right) = 1 + x$ and $r\left(A_{\left\lfloor \frac{r(A)}{3} \right\rfloor + 1}\right) = 2$. This gives $l(A) \leqslant \left\lfloor \frac{r(A)}{3} \right\rfloor + 1 = \left\lceil \frac{r(A)}{3} \right\rceil$. Now factor A as $A_1 \cdots A_s$ where each A_i is an atom of S. Since $r(A_i) \leqslant 3$ for each i, $r(A) \leqslant 3s$. With $s = l(A)$, $\frac{r(A)}{3} \leqslant l(A)$. Since $l(A) \in \mathbb{Z}$, $l(A) \geqslant \left\lceil \frac{r(A)}{3} \right\rceil$.

We now have that

$$\rho(A) = \frac{\left\lfloor \frac{r(A)}{2} \right\rfloor}{\left\lceil \frac{r(A)}{3} \right\rceil} \leqslant \frac{\frac{r(A)}{2}}{\frac{r(A)}{3}} = \frac{3}{2}.$$

Whenever $6|r(A)$ this elasticity is achieved and hence $\rho(S) = 3/2$.

Write $A = A_1A_2 \cdots A_s$ where each A_i is an atom of S and such that $s > l(A)$. That is, $s \geqslant \left\lceil \frac{r(A)}{3} \right\rceil + 1$. Let x denote the number of A_i with $r(A_i) = 2$. Then $r(A) = \sum_{i=1}^{s} r(A_i) = 2x + 3(s - x) = 3s - x$ and hence $s \geqslant \left\lceil \frac{r(A)}{3} \right\rceil + 1 = \left\lceil \frac{3s-x}{3} \right\rceil + 1 = s + 1 + \left\lceil \frac{-x}{3} \right\rceil$. Therefore, $-1 \geqslant \left\lceil \frac{-x}{3} \right\rceil$ and so $x \geqslant 3$. Without loss of generality, assume that $r(A_1) = r(A_2) = r(A_3) = 2$. Then $r(A_1A_2A_3) = 3 + 3$ and hence there are atoms $B_1,\ B_2$ in S with $A_1A_2A_3 = B_1B_2$. Therefore, $A = B_1B_2A_4 \cdots A_s$ is the product of $s - 1$ atoms of S. As this holds for any non-atom $A \in S$, $\Delta(S) = \{1\}$. □

The following is an immediate corollary to the results of this section. Lemma 2.2 provides the transfer homomorphism $\det : S \to \mathbb{Z}$.

Corollary 3.7. *Let S denote the subsemigroup of all matrices in either $M_n(\mathbb{Z})$ or $T_n(\mathbb{Z})$ with composite determinant and let A be a non-atom of S. Then*

1. $L(A) = \left\lfloor \frac{r(A)}{2} \right\rfloor$,
2. $l(A) = \left\lceil \frac{r(A)}{3} \right\rceil$,
3. $\rho(S) = \frac{3}{2}$, *and*
4. $\Delta(S) = 1$.

4. Subsemigroups of $T_n(\mathbb{Z})$

Upper triangular matrices are well-studied, in part because their determinants are easy to compute and because any integer valued matrix can be put in Hermite Normal Form. Moreover, being able to factor matrices in $T_n(\mathbb{Z})$ plays an important role in the Post correspondence problem (cf. [16,11]). In this section we investigate factorization properties of certain subsemiroups of $T_n(\mathbb{Z})$. In particular, we consider $T_2(\mathbb{N})$, $T_2(\mathbb{N}_0)$, $T_n(k\mathbb{N})$ for $k > 1$, and unitriangular matrices – elements of $T_n(\mathbb{N}_0)$ whose diagonal elements are all 1's.

Remark 4.1. Note that each of the results in this section pertaining to subsemigroups of $T_n(\mathbb{Z})$ of upper triangular matrices can be restated for subsemigroups of $L_n(\mathbb{Z})$ of lower triangular matrices.

4.1. $T_n(k\mathbb{Z})$

In this section we take S to be the subsemigroup of all nonzero upper triangular $n \times n$ integer valued matrices whose entries are a multiple of some integer $k > 1$. Since S does not contain the identity matrix, S contains no units. Recall that if x is an integer, divisible by k, then $\nu_k(x)$ is the largest integer t such that $k^t | x$.

Lemma 4.2. *Let $k > 1$ be an integer and let S denote the semigroup of all nonzero matrices in $T_n(k\mathbb{Z})$. Let $A \in S$:*

1. $L(A) = \nu_k(\gcd(A))$.
2. *A is an atom of S if and only if $\nu_k(\gcd(A)) = 1$.*

Proof. First note that if k^u divides all entries of an $n \times n$ matrix B and k^v divides all entries of an $n \times n$ matrix C, then k^{u+v} divides all the entries of their product BC. Thus, if $A = A_1A_2 \cdots A_t$ with each A_i an element of S, then k^t divides all entries of A and so $t \leqslant \nu_k(\gcd(A))$. That is, taking each A_i to be an atom of S, $L(A) \leqslant \nu_k(\gcd(A))$. Note that $kI_n \in S$ and that $A = (kI_n)^{\nu_k(\gcd(A))-1}A'$ for some $A' \in S$. Therefore, $L(A) \geqslant \nu_k(\gcd(A))$ proving (1).

Since A is an atom of S if and only if $L(A) = 1$, (2) follows immediately. $\square$

Theorem 4.3. *Let $k > 1$ be an integer and let S denote the semigroup of all nonzero matrices in $T_n(k\mathbb{Z})$ where $n \geqslant 2$. Then S is bifurcus.*

Proof. Write A as $A = [\mathbf{u} \quad B]$ where $\mathbf{u}$ is a column vector and B is an $n \times (n-1)$ matrix. If $\nu = \nu_k(\gcd(\mathbf{u}))$, then we can factor A as

$$A = [\mathbf{u} \quad B] = \left[\left(\frac{1}{k^{\nu-1}}\right)\mathbf{u} \quad \frac{1}{k}B\right]\begin{bmatrix} k^{\nu-1} & 0 \\ 0 & kI_{n-1} \end{bmatrix},$$

where $\begin{bmatrix} k^{\nu-1} & 0 \\ 0 & kI_{n-1} \end{bmatrix}$ is a 4×4 block matrix with the non-diagonal blocks consisting of only zero entries. By Lemma 4.2, each of these factors is an atom of S and hence S is bifurcus. $\square$

4.2. Unitriangular matrices

In this section we consider S to be the subsemigroup of $T_n(\mathbb{N}_0)$ with ones along the diagonal – that is the set of unitriangular matrices with nonnegative entries. We restrict to nonnegative entries since every unitriangular matrix over the integers is a unit thus making factorization questions trivial. In [9], it was shown that the group of $n \times n$ upper triangular matrix over a semiring is the semidirect product of the group of diagonal matrices and the monoid of unitriangular matrices. The complexity of the set of $n \times n$ unitriangular matrices over a finite field was studied in [15]. Thus we have motivation to study factorization properties in the family of unitriangular matrices.

We will first consider the entire semigroup and then restrict to the subsemigroup consisting only of matrices with nonzero entries above the diagonal. For a matrix $A \in S$, let $\Sigma(A)$ denote the sum of the off diagonal entries in A. Note that since matrices in S contain no negative entries, if $A, B \in S$, then $\Sigma(AB) \geqslant \Sigma(A) + \Sigma(B)$. Therefore, the only unit of S is the $n \times n$ identity matrix I_n. Indeed, if $AB = I_n$, then $0 \leqslant \Sigma(A) \leqslant \Sigma(AB) = \Sigma(I_n) = 0$ and hence $\Sigma(A) = 0$.

For i and j with $i < j$, let $E_{i,j} \in M_n(\mathbb{N}_0)$ denote the matrix whose only nonzero entry is $e_{i,j} = 1$. We note that for each pair i, j, $E_{i,j}$ is not an element of S, but $I + E_{i,j}$ is an element of S.

Theorem 4.4. *Let S denote the subsemigroup of $T_n(\mathbb{N}_0)$ of unitriangular matrices and let $A \in S$. Then $L(A) = \Sigma(A)$.*

Proof. Suppose that $A = A_1A_2 \cdots A_t$ where each A_i is a nonunit of S ($A_i \neq I_n \forall i$). Then $\Sigma(A) \geqslant t$ and hence $L(A) \leqslant \Sigma(A)$. We now show, via induction on $\Sigma(A)$, that $L(A) \geqslant \Sigma(A)$. If $\Sigma(A) = 1$, then clearly A is irreducible and hence $L(A) = 1$. Now suppose that for some $m \geqslant 1$, whenever $\Sigma(A) \leqslant m$, then $L(A) \geqslant \Sigma(A)$.

If $\Sigma(A) = m + 1$, then either $a_{i,j} = m + 1$ for some distinct i, j or there are distinct off diagonal entries of A which are nonzero. If $a_{i,j} = m + 1$ for some pair i, j, then $A = I + (m + 1)E_{i,j} = (I + E_{i,j})^{m+1}$ and hence $L(A) \geqslant m + 1$. If $\Sigma(A) = m + 1$ and there are distinct off diagonal entries of A which are nonzero, consider $i + j$ maximal among the non-diagonal elements $a_{i,j} > 0$. Then we can factor A as $A = (A - a_{i,j}E_{i,j})(I + E_{i,j})^{a_{i,j}}$. Clearly $\Sigma(A - a_{i,j}E_{i,j}) = (m + 1) - a_{i,j}$ and by our induction hypothesis, $L(A - a_{i,j}E_{i,j}) \geqslant (m + 1) - a_{i,j}$. Moreover, $L((I + E_{i,j})^{a_{i,j}}) \geqslant a_{i,j}$ and hence $L(A) \geqslant (m + 1) - a_{i,j} + a_{i,j} = m + 1$. $\square$

The following result, which follows trivially from Theorem 4.4, gives a classification of the atoms of S.

Corollary 4.5. *Let S denote the semigroup of $T_n(\mathbb{N}_0)$ of unitriangular matrices and let $A \in S$. Then A is an atom of S if and only if $\Sigma(A) = 1$.*

Theorem 4.6. *Let S denote the semigroup of $T_n(\mathbb{N}_0)$ of unitriangular matrices:*

1. *If $n = 2$, then S is factorial.*
2. *If $n \geqslant 3$, then $\rho(S) = \infty$.*

Proof. If $n = 2$, then the only unit in S is I_2 and the only atom of S is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $B \in S$ is a nonunit, then $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ with $b \geqslant 1$. The only factorization of B is $B = A^b$ and hence S is factorial.

Now suppose $n \geqslant 3$ and write $A = (I + E_{1,2})^a (I + E_{2,3})^a = I + aE_{1,2} + aE_{2,3} + a^2 E_{1,3}$. Thus $l(A) = 2a$. Moreover, $L(A) = 2a + a^2$ and hence $\rho(S) \geqslant \lim_{a\to\infty} \rho(A) \geqslant \lim_{a\to\infty} \frac{2a+a^2}{2a} = \infty$. $\square$

We conclude this section by considering the subsemigroup S of $T_n(\mathbb{N})$ of unitriangular matrices. Note that we now require all entries above the diagonal to be positive. We now show that S is bifurcus and hence has infinite elasticity.

Theorem 4.7. *Let $n \geqslant 4$ and let S denote the semigroup of $T_n(\mathbb{N})$ of unitriangular matrices. Then S is bifurcus.*

Proof. First note that if $X \in S$ can be factored as the product of two elements of S, then each of the superdiagonal entries of X is at least two. Consequently, if any of the superdiagonal entries of X are one, then X is an atom of S. Now, suppose that $A \in S$ can be written as the product of two elements $B, C \in S$. That is,

$$A = \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{n-1,n} \\ 0 & \cdots & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & b_{1,2} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & b_{n-1,n} \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & c_{1,2} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & c_{n-1,n} \\ 0 & \cdots & \cdots & 1 \end{bmatrix} = BC.$$

Let $U = I + (1 - b_{1,2})E_{1,2} + c_{n-1,n}E_{n-1,n}$ and $V = I + (b_{1,2} - 1)E_{1,2} + (1 - c_{n-1,n})E_{n-1,n}$; that is,

$$U = \begin{bmatrix} 1 & 1 - b_{1,2} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1 & c_{n-1,n} - 1 \\ 0 & \cdots & & \cdots & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & b_{1,2} - 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1 & 1 - c_{n-1,n} \\ 0 & \cdots & & \cdots & 1 \end{bmatrix}.$$

Note that U and V are not elements of S. However, since $UV = I_n$, $A = BC = B(UV)C = (BU)(VC)$. Moreover, $BU, VC \in S$ are atoms since they each possess a one as a superdiagonal entry. Thus A can be factored as the product of two atoms of S and hence S is bifurcus. $\square$

4.3. $T_2(\mathbb{N})$

Throughout this section S will denote the semigroup of all 2×2 upper triangular matrices with positive integer entries; that is, if $A \in S$, then $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for some positive integers a, b and c. Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$, S contains no units. The following lemma classifies all atoms of S.

Lemma 4.8. *Let* $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in S$*. Then A is an atom of S if and only if* $b = 1$*.*

Proof. If $b > 1$ then we can write $b = m + n$ with $m, n \geqslant 1$. Thus we can factor A as

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & c \end{bmatrix} \begin{bmatrix} a & n \\ 0 & 1 \end{bmatrix}$$

and hence A is a non-atom. Conversely, suppose that A can be written as the product of two elements of S as

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} s & m \\ 0 & u \end{bmatrix} \begin{bmatrix} t & n \\ 0 & v \end{bmatrix}.$$

Then $b = sn + mv > 1$. □

Theorem 4.9. *Let S be the semigroup of all* 2×2 *upper triangular matrices with positive integer entries and let* $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in S$*. Then*

1. $L(A) = b$.
2. $\rho(S) = \infty$.
3. *For every prime p,* $p - 1 \in \Delta(S)$*.*

Proof. Consider $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in S$. If $A = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \cdots \begin{bmatrix} a_s & b_s \\ 0 & c_s \end{bmatrix}$, then $b \geqslant b_1 + b_2 + \cdots + b_s$ and hence $s \leqslant b$. That is, $L(A) \leqslant b$. If $b = 1$, then A is an atom by Lemma 4.8 $k = 1$ and thus has a factorization of length $b = 1$. If $b \geqslant 2$, then we can factor A as

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{b-2} \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}$$

and hence (again by Lemma 4.8) A has a factorization of length b.

Let n be a positive integer and consider the following matrix $B = \begin{bmatrix} 2^n & 2^{n+1} \\ 0 & 2^n \end{bmatrix} \in S$. Since $B = \begin{bmatrix} 2^n & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2^n \end{bmatrix}$ $l(B) = 2$ by Lemma 4.8. From (1), $L(B) = 2^{n+1}$. Therefore, $\rho(B) = \frac{2^{n+1}}{2}$. Taking the limit as n approaches infinity gives us that $\rho(S) = \infty$.

Let p be a prime and consider the matrix $P = \begin{bmatrix} p & p+1 \\ 0 & 1 \end{bmatrix} \in S$. Since p is prime, if

$$P = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix},$$

then $a_1, a_2 \in \{1, p\}$ and $c_1 = c_2 = 1$. Thus, by Lemma 4.8, the only atoms that can appear in a factorization of P are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} p & 1 \\ 0 & 1 \end{bmatrix}$. Furthermore, the matrix $\begin{bmatrix} p & 1 \\ 0 & 1 \end{bmatrix}$ must appear exactly once in any factorization of P. Thus, the only factorizations of P have the form

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^m \begin{bmatrix} p & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{t-m-1},$$

where t is the length of the factorization and $0 \leqslant m \leqslant t - 1 \leqslant p$. If $t - m - 1 = 0$, then $t = p + 1$. If $t - m - 1 \geqslant 1$, then

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^m \begin{bmatrix} p & p(t-m-1)+1 \\ 0 & 1 \end{bmatrix}$$

and hence $p(t - m - 1) + 1 \leqslant p + 1$ which implies that $t - m - 1 = 1$; *i.e.*, $t = 2$. Therefore, all factorizations of P have length 2 or $p + 1$ and hence $\Delta(P) = \{p - 1\}$. □

4.4. $T_2(\mathbb{N}_0)$

In this section we consider the semigroup S of upper triangular 2×2 matrices with nonnegative entries and non-zero determinant; that is, if $A \in S$ then $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a, c \in \mathbb{N}$ and $b \in \mathbb{N}_0$. Note that in this case $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the only unit of S. As in the previous section, we first classify the atoms of S.

Lemma 4.10. *Let S denote the subsemigroup of matrices in $T_2(\mathbb{N}_0)$ with nonzero determinant. The set of atoms of S consists of the matrix $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the matrices – for each prime p – $Y_p = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ and $Z_p = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$.*

Proof. Suppose that $X = X_1X_2$ for some $X_1, X_2 \in S$. Since $\det(X) = 1$, $\det(X_1) = \det(X_2) = 1$ and we can write

$$X_1X_2 = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

where $m + n = 1$. Thus either X_1 or X_2 is the identity and hence X is an atom.

Suppose now that p is prime and $Y_p = Y_1Y_2$ for some $Y_1, Y_2 \in S$. Since p is prime, either

$$Y_1Y_2 = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & p \end{bmatrix}$$

with $0 = n + mp$ or

$$Y_1Y_2 = \begin{bmatrix} 1 & n \\ 0 & p \end{bmatrix}\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

with $0 = n + m$. In either case, $m = 0$ and hence Y_p is an atom of S. Similarly one can see that each Z_p is an atom of S.

Finally, we show that these are the only atoms of S. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in S$. Following the proof of Lemma 4.8, we see that if $b \geqslant 2$ then A is a non-atom of S. Suppose $A = \begin{bmatrix} a & 1 \\ 0 & c \end{bmatrix}$ and note that we can factor A as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus A is an atom if and only if $a = c = 1$; that is, $A = X$. Now suppose $A = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ and note that A can be factored as

$$A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & 1 \end{bmatrix},$$

where a_1 and a_2 are factors of a and c_1 and c_2 are factors of c. Thus A is an atom if and only if $A = Y_p$ or $A = Z_p$ for some prime p. $\square$

Throughout the remainder of this section we let $r(A)$ denote the number of (not necessarily distinct) prime factors of $\det(A)$.

Lemma 4.11. *If A can be factored as $A = A_1A_2 \cdots A_t$ with each A_i an atom of S, then $t = r(A) + k$ where $k = \left|\left\{i : A_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}\right|$.*

Proof. For each i, A_i is an atom and thus $\det(A_i)$ is either 1 or is prime. Since $\det(A) = \det(A_1)\det(A_2)\cdots\det(A_t)$, $|\{i : \det(A_i) \text{ is prime}\}| = r(A)$. If $k = |\{i : \det(A_i) = 1\}|$ then the length of the factorization is

$$t = |\{i : \det(A_i) \text{ is prime}\}| + |\{i : \det(A_i) = 1\}| = r(A) + k. \qquad \square$$

Theorem 4.12. *Let S denote the subsemigroup of invertible matrices in $T_2(\mathbb{N}_0)$ and let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in S$:*

1. *If $b = 0$, then $l(A) = r(A)$.*
2. *If $b|ac$, then $l(A) = r(A) + 1$.*
3. *$L(A) = r(A) + b$.*

Proof. Suppose that $b = 0$ and write $l(A) = r(A) + k$ as in Lemma 4.11. If $k \geqslant 1$ then any factor containing $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ cannot have two nonzero entries. This contradicts $b = 0$ and hence $k = 0$. That is, $l(A) = r(A)$.

Now suppose that $b|ac$. Again write $l(A) = r(A) + k$ as in Lemma 4.11. Since $b > 0$, $k \geqslant 1$ and hence $l(A) \geqslant r(A) + 1$. Write $a = ma'$, $c = nc'$ and $b = a'c'$ and factor A as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}\begin{bmatrix} a' & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c' \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} l(A) &\leqslant l\left(\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}\right) + l\left(\begin{bmatrix} a' & 0 \\ 0 & 1 \end{bmatrix}\right) + l\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) + l\left(\begin{bmatrix} 1 & 0 \\ 0 & c' \end{bmatrix}\right) + l\left(\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= r(n) + r(a') + 1 + r(c') + r(m) \\ &= r(na'c'm) + 1 \\ &= r(A) + 1. \end{aligned}$$

Note that for any matrix $\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \in S$,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^r \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^s = \begin{bmatrix} a' & d \\ 0 & c' \end{bmatrix}$$

with $d \geqslant r + s$. Thus, if $A = A_1A_2\cdots A_t$ with each A_i an atom of S, $b \geqslant |\{i : \det(A_i) = 1\}|$. By Lemma 4.11, $l(A) = r(A) + k$ with $b \geqslant k$. Thus $L(A) \leqslant r(A) + b$. Since

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^b \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix},$$

we have that

$$\begin{aligned} L(A) &\geqslant l\left(\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\right) + b \cdot l\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) + l\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= r(c) + b + r(a) \\ &= r(A) + b. \qquad \square \end{aligned}$$

Corollary 4.13. *Let S denote the subsemigroup of invertible matrices in $T_2(\mathbb{N}_0)$ and let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in S$:*

1. *If $b = 0$, then $\rho(A) = 1$.*
2. *If $b|ac$, then $\rho(A) = \frac{r(A)+b}{r(A)+1}$.*
3. *$\rho(S) = \infty$.*

Proof. The proofs of (1) and (2) follow immediately from Theorem 4.12.

Consider, for each prime p the matrix $A_p = \begin{bmatrix} p & p \\ 0 & p \end{bmatrix}$. By (2), $\rho(A_p) = \frac{p+2}{3}$. Allowing p to tend towards infinity, we see that $\rho(S) = \infty$. $\square$

5. Gaussian matrices

In this section we investigate the factorization properties of semigroups of $n \times n$ Gaussian matrices over an additive subsemigroup R of $\mathbb{N}_0$ – matrices differing from the identity matrix I_n only in the jth column which has the form

$$[0 \quad \cdots \quad 0 \quad 1 \quad a_{j+1,j} \quad a_{j+2,j} \quad \cdots \quad a_{n,j}]^T$$

for some j, $1 \leqslant j \leqslant n - 1$. Gaussian elimination of column j of a matrix is achieved by multiplication by a Gaussian matrix with nonzero off diagonal entries in the jth column. This type of matrix is essential in LU factorizations and thus have a wide array of applications in both pure and applied mathematics (cf. [8,17,6]).

Note that when two Gaussian matrices are multiplied, the only entries affected are the off diagonal entries. Furthermore, if $A = BC$ where A, B, C are Gaussian matrices with all nonzero off diagonal entries in fixed column j, then

$$\begin{bmatrix} a_{j+1,j} \\ a_{j+2,j} \\ \vdots \\ a_{n,j} \end{bmatrix} = \begin{bmatrix} b_{j+1,j} \\ b_{j+2,j} \\ \vdots \\ b_{n,j} \end{bmatrix} + \begin{bmatrix} c_{j+1,j} \\ c_{j+2,j} \\ \vdots \\ c_{n,j} \end{bmatrix}.$$

This immediately yields the following lemma.

Proposition 5.1. *Let S denote the semigroup of $n \times n$ Gaussian matrices over an additive subsemigroup R of $\mathbb{N}_0$ differing from the identity matrix I_n only in the jth column which has the form $[0 \ \cdots \ 0 \ 1 \ a_{j+1,j} \ a_{j+2,j} \ \cdots \ a_{n,j}]^T$ (where each $a_{i,j} \in R$) for some j, $1 \leqslant j \leqslant n - 1$. Finally, set $T = R^{n-j} \cup \{0\}$. Then there exists a transfer homomorphism $\phi : (S, \cdot) \to (T, +)$ defined by $[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] \mapsto [a_{j+1,j} \ a_{j+2,j} \ \cdots \ a_{n,j}]^T$. In particular, S is commutative.*

We note that the results of Proposition 5.1 hold for the semigroups such as $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{N}_{\geqslant k} \cup \{0\}$ for any $k \in \mathbb{N}$ and $k\mathbb{N}$ for any $k \in \mathbb{N}$. The following two theorems give factorization properties of Gaussian matrices over $\mathbb{N}$ and over $\mathbb{N}_0$ in which we see a dramatic difference in uniqueness of factorization depending on whether or not entries of the jth column below the diagonal are allowed to be zero.

Theorem 5.2. *Let S denote the subsemigroup of $M_n(\mathbb{N})$ of Gaussian matrices with nonzero off diagonal entries in column j and let $A \in S$:*

1. *A is an atom if and only if 1 is an off diagonal entry of A.*
2. *$L(A)$ is equal to the minimal positive off diagonal entry of A.*
3. *If $j = n - 1$, then S is factorial.*
4. *If $j \neq n - 1$, then S is bifurcus.*

Proof. In light of Proposition 5.1 we consider, instead, factorizations in $T = \{0\} \cup \mathbb{N}^{n-j}$. By Geroldinger and Halter-Koch [7], $\mathbb{N}^k \cup \{0\}$ is a finitely primary monoid of rank k for each k and thus the results follow immediately. $\square$

Theorem 5.3. *Let S denote the subsemigroup of $M_n(\mathbb{N}_0)$ of Gaussian matrices with nonnegative off diagonal entries in column j and let $A \in S$:*

1. *A is an atom if and only if all entries in column j are zero except for* $a_{j,j} = a_{k,j} = 1$ *for some* $k \in \{j+1, j+2, \ldots, n\}$.
2. *S is factorial*.

Proof. Again, in light of Proposition 5.1, we consider factorization in $T = \mathbb{N}_0^{n-j}$. Again appealing to [7], we have that $\mathbb{N}_0^k$ is factorial and hence the result follows immediately. □

6. Rank one matrices

Rank one matrices have been intensely studied from many different perspectives and have a wide array of applications (cf. [19,21,20]). Throughout this section, let $n \geqslant 2$ and let S denote the subsemigroup of $M_n(\mathbb{N})$ of matrices with rank one. Note that S contains no identity and no units. We recall that if $A \in M_n(\mathbb{N})$ has rank one, then $A = \mathbf{u}\mathbf{v}^T$ for some vectors $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$. The following lemma will be useful in studying factorizations in S.

6.1. Preliminaries

Lemma 6.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{N}^n$ *and set* $A = \mathbf{u}\mathbf{v}^T$. *Then* $\gcd(A) = \gcd(\mathbf{u})\gcd(\mathbf{v})$.

Proof. Note that $\frac{1}{\gcd(\mathbf{u})\gcd(\mathbf{v})}A = \left(\frac{1}{\gcd(\mathbf{u})}\mathbf{u}\right)\left(\frac{1}{\gcd(\mathbf{v})}\mathbf{v}^T\right)$ and thus we may assume that $\gcd(\mathbf{u}) = \gcd(\mathbf{v}) = 1$. Suppose that $m|\gcd(A)$ for some $m > 1$. Then m divides each entry of A and in particular each entry in column i of A for each $i \in \{1, 2, \ldots, n\}$. However, the ith column of A is the vector $v_i\mathbf{u}$. Since $\gcd(\mathbf{u}) = 1$, $m|v_i$. As this holds for all $i \in \{1, 2, \ldots, n\}$, $m|\gcd(\mathbf{v})$, a contradiction. Thus $\gcd(A) = 1$. □

Theorem 6.2. *Let S denote the subsemigroup of* $M_n(\mathbb{N})$ *consisting of all matrices with rank one and let* $A \in S$:

1. *A is an atom of S if and only if* $\gcd(A) < n$.
2. *S is bifurcus*.

Proof. Suppose that $A = A_1A_2$ for $A_1, A_2 \in S$. Since A_1 and A_2 have rank one, there exist vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{N}^n$ such that $A_1 = \mathbf{u}_1\mathbf{v}_1^T$ and $A_2 = \mathbf{u}_2\mathbf{v}_2^T$. Since $\mathbf{v}_1^T\mathbf{u}_2$ is a scalar, we have that

$$A = (\mathbf{u}_1\mathbf{v}_1^T)(\mathbf{u}_2\mathbf{v}_2^T) = \mathbf{u}_1(\mathbf{v}_1^T\mathbf{u}_2)\mathbf{v}_2^T = (\mathbf{v}_1^T\mathbf{u}_2)(\mathbf{u}_1\mathbf{v}_2^T).$$

Since the entries of $\mathbf{v}_1^T$ and $\mathbf{u}_2$ are positive integers, $\gcd(A) \geqslant (\mathbf{v}_1^T\mathbf{u}_2) \geqslant n$. Now suppose that $\gcd(A) \geqslant n$ and write $A = \gcd(A)B$ where B is an atom of S. Write $B = \mathbf{u}\mathbf{v}^T$ for some vectors $\mathbf{u}$ and $\mathbf{v}$. Since $\gcd(A) \geqslant n$, write $\gcd(A) = \mathbf{x}^T\mathbf{y}$ where $\mathbf{x} = \left[(\gcd(A) - n + 1)\ 1\ 1\ \cdots\ 1\right]^T$ and $\mathbf{y} = \left[1\ 1\ \cdots\ 1\right]^T$. Then

$$A = \gcd(A)B = (\mathbf{x}^T\mathbf{y})(\mathbf{u}\mathbf{v}^T) = \mathbf{u}(\mathbf{x}^T\mathbf{y})\mathbf{v}^T = (\mathbf{u}\mathbf{x}^T)(\mathbf{y}\mathbf{v}^T).$$

Since $B = \mathbf{u}\mathbf{v}^T$ and $\gcd(B) = 1$, $\gcd(\mathbf{u}) = \gcd(\mathbf{v}) = 1$ by Lemma 6.1. Again applying Lemma 6.1, we see that $\mathbf{u}\mathbf{x}^T$ and $\mathbf{y}\mathbf{v}^T$ are atoms of S. Thus A is an atom of S if and only if $\gcd(A) = 1$. Moreover, we have shown that S is bifurcus as every non-atom can be factored as the product of two atoms of S. □

The following theorem gives a formula for $L(A)$ where $A \in S$ and shows that computing $L(A)$ is an NP-complete problem. First, we define, for integers n and m, $\Psi_n(m)$ to be the greatest integer k such that there exist integers $m_1, m_2, \ldots, m_k, r$ such that $m = m_1m_2\cdots m_kr$ with $r < n \leqslant m_i$ for each i. We note that $\Psi_n(m)$ is at most the number of primes counting multiplicity in the prime factorization of m, with equality if n=2.

Theorem 6.3. *Let S denote the subsemigroup of $M_n(\mathbb{N})$ consisting of all matrices with rank one and let $A \in S$:*

1. *Then $L(A) = \Psi_n(\gcd(A)) + 1$.*
2. *Calculating $L(A)$ is a NP-complete problem.*

Proof. Assume that A has some factorization of length t and write $A = \prod_{i=1}^{t} A_i = \prod_{i=1}^{t} \mathbf{u}_i\mathbf{v}_i^T$ where each $A_i = \mathbf{u}_i\mathbf{v}_i^T$ is an atom of S. We can rewrite this product as

$$A = \prod_{i=1}^{t} \mathbf{u}_i\mathbf{v}_i^T = \mathbf{u}_1\mathbf{v}_t^T \prod_{i=1}^{t-1} \langle \mathbf{v}_i, \mathbf{u}_{i+1} \rangle,$$

where $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}\mathbf{y}^T$ denotes the standard inner product of vectors $\mathbf{x}$ and $\mathbf{y}$. Since each $\langle \mathbf{v}_i, \mathbf{u}_{i+1} \rangle$ is the sum of n positive integers, $\langle \mathbf{v}_i, \mathbf{u}_{i+1} \rangle \geqslant n$ for each $i \in \{1, 2, \ldots, t-1\}$. By Lemma 6.1, $\gcd(A) = \gcd(\mathbf{u}_1)\gcd(\mathbf{v}_t)\prod_{i=1}^{t-1}\langle \mathbf{v}_i, \mathbf{u}_{i+1} \rangle$ and hence

$$\Psi_n(\gcd(A)) = \Psi_n\left(\gcd(\mathbf{u}_1)\gcd(\mathbf{v}_t)\prod_{i=1}^{t-1}\langle \mathbf{v}_i, \mathbf{u}_{i+1} \rangle\right) \geqslant \Psi_n\left(\prod_{i=1}^{t-1}\langle \mathbf{v}_i, \mathbf{u}_{i+1} \rangle\right) \geqslant t-1.$$

Therefore, $L(A) \leqslant \Psi_n(\gcd(A)) + 1$.

Now suppose that $\Psi_n(A) = k$ and write $\gcd(A) = rm_1m_2 \cdots m_k$ for some integers $r, m_1, m_2, \ldots, m_k$ with $r < n \leqslant m_i$ for each $i \in \{1, 2, \ldots k\}$. For each $i \in \{1, 2, \ldots, k\}$ we can write $A = rm_1m_2 \cdots m_iB_i$ for an appropriate matrix $B_i \in S$. Since $\gcd(m_iB_i) \geqslant m_i \geqslant n$, m_iB_i is not an atom of S and thus we can write $m_iB_i = B_{i-1}C_i$ for some atom $C_i \in S$. Therefore,

$$A = rm_1m_2 \cdots m_kB_k = (rB_0)C_1C_2 \cdots C_k.$$

Therefore, A has a factorization of length at least $k+1$ and consequently $L(A) \geqslant \Psi_n(\gcd(A)) + 1$.

If $m = p_1p_2 \cdots p_t$ with each p_i prime, then $\Psi_n(m)$ is equal to the maximum number of disjoint subsets $\{a_1, a_2, \ldots, a_s\}$ of $\{1, 2, \ldots, t\}$ such that $p_{a_1}p_{a_2} \cdots p_{a_s} \geqslant n$. Thus $\log(p_{a_1}) + \log(p_{a_2}) + \cdots + \log(p_{a_s}) \geqslant \log(n)$ and hence finding $\Psi_n(m)$ is equivalent to solving a bin-covering problem. Therefore, calculating $\Psi_n(m)$ is a NP-complete problem. □

6.2. Rank one matrices with entries in $m\mathbb{N}$

In this section we restrict to the subsemigroup S_m of rank one matrices in $M_n(m\mathbb{N})$ where $m \geqslant 1$ is an integer. The following lemma allows us to apply factorization information gleaned from S_1 (Theorem 6.2) to the semigroup S_m.

Lemma 6.4. *Let $m \geqslant 1$ be an integer, let S_m denote the subsemigroup of rank one matrices in $M_n(m\mathbb{N})$ and let $A \in S_m$:*

1. *If $m^2 \nmid \gcd(A)$ then A is an atom of S_m.*
2. *If $A = m^2B$ with $B \in S_1$, then A is an atom of S_m if and only if B is an atom of S_1.*

Proof. Suppose that $A = A_1A_2$ with $A_1, A_2 \in S_m$. Then $A = (mB_1)(mB_2) = m^2B_1B_2$ for some $B_1, B_2 \in S_1$. Thus if $m^2 \nmid \gcd(A)$, then A is an atom of S_m.

Now suppose that $A = m^2B$ for some $B \in S_1$. If $B = B_1B_2$ for some $B_1, B_2 \in S_1$ then $A = m^2B = (mB_1)(mB_2)$ with $mB_1, mB_2 \in S_m$. If $A = A_1A_2$ with $A_1, A_2 \in S_m$ then $m^2B = A = A_1A_2 = (mC_1)(mC_2)$ for some $C_1, C_2 \in S_1$ and hence $B = C_1C_2$. Thus A is an atom of S_m if and only if B is an atom of S_1. □

Theorem 6.5. *Let $m \geqslant 1$ be an integer and let S_m denote the subsemigroup of rank one matrices in $M_n(m\mathbb{N})$. Then S_m is bifurcus.*

Proof. Let $A \in S_m$ be a non-atom and factor A as $A = A_1A_2$ for some $A_1, A_2 \in S_m$. By Lemma 6.4, $A = (mB_1)(mB_2)$ for some $B_1, B_2 \in S_1$. By Theorem 6.2, S_1 is bifurcus and so $A = m^2B_1B_2 = m^2P_1P_2 = (mP_1)(mP_2)$ for some atoms $P_1, P_2 \in S_1$. If $mP_i = XY$ for $X, Y \in S_m$ then $mP_i = mX'Y$ with $X' \in S_1$ and hence $P_i = X'Y$ with $X', Y \in S_1$, contradicting that P_i is an atom of S_1. Therefore, A can be factored as the product of two atoms in S_m and hence S_m is bifurcus. $\square$

6.3. The semiring of single-valued matrices

In this section we consider a specific class of rank one matrices – the class of $n \times n$ matrices in which all entries are identical. Jacobson [13] considered the 2×2 case and was able to classify the atoms of this semigroup and give examples of nonunique factorization. We extend these results to $n \times n$ single-valued matrices over $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$. We adopt Jacobson's notation and let $[a]$ denote the $n \times n$ matrix for which every entry is a. The simple observation that if $[a]$ and $[b]$ are $n \times n$ matrices, then $[a][b] = [nab]$ allows us to classify the atoms and to compute various factorization invariants. The following theorems are stated for the semigroup S of nonzero single-valued matrices with integer entries, but the results hold if we restrict to nonnegative entries or to only positive entries. In either case, it is easy to see that S has no units.

Lemma 6.6. *Let S denote the semigroup of nonzero single-valued $n \times n$ matrices with integer entries and let $[a] \in S$. Then $[a]$ is an atom of S if and only if $n \nmid a$.*

Proof. If $[a]$ factors as $[a] = [s][t]$, then $[a] = [s][t] = [nst]$ and hence $n|a$. Conversely, if $n|a$ then we can factor $[a]$ as $[a] = [nst] = [s][t]$. $\square$

Recall that $\nu_n(a)$ denotes the largest integer k such that a is divisible by n^k.

Theorem 6.7. *Let $n \geqslant 2$ and let S denote the semigroup of nonzero single-valued $n \times n$ matrices with integer entries. Let $[a] \in S$:*

1. *$L([a]) = \nu_n(a) + 1$.*
2. *If $n = p^k$ for some prime p, then $l([a]) = \left\lceil \frac{\nu_p(a)+k}{2k-1} \right\rceil$ and $\rho(S) = \frac{2k-1}{k}$.*
3. *If n is not the power of a prime, then S is bifurcus.*
4. *If n is prime, then S is half-factorial.*
5. *If n is not prime, then $\Delta(S) = \{1\}$.*

Proof. Suppose that $[a]$ can be factored as $[a] = [a_1][a_2]\cdots[a_t]$ with each $[a_i] \in S$. Then $[a] = [n^{t-1}a_1a_2\cdots a_t]$ and hence $\nu_n(a) \geqslant t - 1$. Therefore, $L([a]) \leqslant \nu_n(a) + 1$. Since $[a]$ can be factored as $[a] = [1]^{\nu_n(a)}\left[\frac{a}{n^{\nu_n(a)}}\right]$, $L([a]) \geqslant \nu_n(a) + 1$.

Assume now that $n = p^k$ for some prime p. Write $a = p^m y$ with $p \nmid y$ and suppose that $[a]$ has a factorization of length d. Then

$$[a] = [p^m y] = [p^{m_1}y_1][p^{m_2}y_2]\cdots[p^{m_d}y_d]$$

where $y = y_1y_2\cdots y_d$ and $m = m_1 + m_2 + \cdots + m_d + k(d-1)$. Since we are assuming each $[p^{m_i}y_i]$ is an atom of S, $0 \leqslant m_i \leqslant k-1$ for each $i \in \{1, 2, \ldots, d\}$. Let c be the nonnegative integer such that $m_1 + m_2 + \cdots + m_d = d(k-1) - c$. Then

$$d = \frac{m - d(k-1) + c}{k} + 1 = \frac{m + c + k}{2k-1} \geqslant \left\lceil \frac{m+k}{2k-1} \right\rceil.$$

Therefore, $l([a]) = \left\lceil \frac{m+k}{2k-1} \right\rceil$.

Observe that for any $[p^m y] \in S$,

$$\rho([p^m y]) = \frac{\left\lfloor \frac{m}{k} \right\rfloor + 1}{\left\lceil \frac{m+k}{2k-1} \right\rceil} \leqslant \frac{\frac{m}{k} + 1}{\frac{m+k}{2k-1}} = \frac{2k-1}{k}.$$

This elasticity is achieved since $[p^{2k(k-1)}] = [1]^{2k-1} = [p^{k-1}]^k$.

If n is prime, then $\rho(S) = 1$ by (2), hence half-factorial. If $n = p^k$ for some $k > 1$ then writing

$$[a] = [p^m y] = [p^{m_1} y_1][p^{m_2} y_2] \cdots [p^{m_d} y_d],$$

where $y = y_1 y_2 \cdots y_d$ and $m = m_1 + m_2 + \cdots + m_d + k(d-1)$ we see that $m_1 + m_2 + \cdots + m_d$ can take all integer values between 0 and $n(k-1)$. Thus, $\Delta([a]) = \{1\}$ for all $[a] \in S$ and hence $\Delta(S) = \{1\}$.

Finally, assume that $n = st$ with $s, t > 1$ and $\gcd(s, t) = 1$. Then if $[a]$ is not an atom of S we can write $[a] = [n^{\nu_n(a)} y]$ where $n \nmid y$. Without loss of generality, assume that $s \nmid y$. Then we can factor $[a]$ as

$$[a] = [n^{\nu_n(a)} y] = [s^{\nu_n(a)} t^{\nu_n(a)} y] = [s^{\nu_n(a)-1}][t^{\nu_n(a)-1} y]$$

and hence $l([a]) = 2$. Therefore, S is bifurcus and $\Delta(S) = \{1\}$. $\square$

7. Open problems

We conclude with a list of open problems:

1. Let $S = T_2(\mathbb{N})$ and let $A \in S$:
   (a) Determine a formula for $l(A)$.
   (b) Determine a formula for $\rho(A)$.
   (c) Calculate $\Delta(S)$.
2. Let $S = T_2(\mathbb{N}_0)$ and let $A \in S$:
   (a) Determine a general formula for $l(A)$.
   (b) Determine a formula for $\rho(A)$.
   (c) Calculate $\Delta(S)$.
3. Let $S = T_n(\mathbb{N})$ for $n > 2$ and let $A \in S$:
   (a) Determine a formula for $l(A)$ and $L(A)$.
   (b) Determine a general formula for $\rho(A)$.
   (c) Calculate $\Delta(S)$ and $\rho(S)$.
4. Let $S = T_n(\mathbb{N}_0)$ for $n > 2$ and let $A \in S$:
   (a) Determine a formula for $l(A)$ and $L(A)$.
   (b) Determine a general formula for $\rho(A)$.
   (c) Calculate $\Delta(S)$ and $\rho(S)$.
5. Let S denote the subsemigroup of $M_n(\mathbb{N}_0)$ of unitriangular matrices and let $A \in S$:
   (a) Determine a formula for $l(A)$.
   (b) Determine a general formula for $\rho(A)$.
   (c) Calculate $\Delta(S)$.

6. Let S denote the subsemigroup of $T_n(\mathbb{N}_0)$ of triangular unitriangular matrices and let $A \in S$:

   (a) Enumerate the atoms of S.
   (b) Determine a formula for $l(A)$ and $L(A)$.
   (c) Determine a general formula for $\rho(A)$.

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