Jump Systems and Laminated Manhattan Sets

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A jump system is a set of lattice points satisfying a certain "two-step" axiom. A Manhattan set is the convex hull of a two-dimensional jump system. Taking multiple Manhattan sets, in layers, forms a three-dimensional object. We determine under what conditions this object is, in turn, a jump system.

1. INTRODUCTION

A *jump system* is a collection of lattice points that obeys the simple axiom below. They model both delta-matroids (hence matroids) and degree sequences of subgraphs, and have been the object of recent interest [1, 2, 3, 5, 6, 7, 9]. For lattice points x, y, z, we say that z is a *step* from x toward (in the direction of) y if $|z - x| = 1$ and $|z - y| < |x - y|$. We denote this by $x \xrightarrow{y} z$.

Definition 1.1. Let J be a set of lattice points. It is a *jump system* if for all lattice points x, y, z where $x \xrightarrow{y} z$, $x, y \in J$, and $z \notin J$, then there is a $z' \in J$ that satisfies $z \xrightarrow{y} z'$.

Observe that translation of J, reflection in one or more axes, and swapping coordinates will preserve the above definition. One-dimensional jump systems are easy to characterize (no gaps of size greater than one). Two-

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dimensional jump systems have been characterized [7, 4]. The convex hull of a two-dimensional jump system, in turn a jump system, is known [7] to be what is called below a Manhattan set. Our main result is a characterization of certain types of three-dimensional jump systems; specifically, those that are laminated Manhattan sets.

2. MANHATTAN POLYGONS AND SETS

A *Manhattan polygon* is a convex polygon whose (nonempty set of) vertices lie on the integer lattice, and whose (possibly empty set of) edges each have slope one of $\{0, 1, -1, \infty\}$. They are exactly integral bisubmodular polyhedra of dimension two [2]. Pictured here are sixteen Manhattan polygons exhibiting varying degrees of symmetry:

We now count how many shapes are possible, where the "shape" of a Manhattan polygon is determined by which of the eight possible edges are present. Starting with all eight corners, we may combine adjacent corners. By this we mean juxtaposing those corners, losing the edge between them. There is a single restriction to this process: if we combine four consecutive corners, then the polygon flattens out, and the other four corners must also be combined.

If we do combine four consecutive corners of the general shape this way, two corners will remain, and the shape will be a line segment. In the table below, this is denoted by $4+4$, since the resulting shape has two corners, each arising from four corners of the general shape. There are four such line segments possible – horizontal, vertical, NE-SW, and NW-SE.

To count Manhattan triangles, we must create three corners out of eight, which can only be done by merging three, three, and two corners of the general shape into the corners of the triangle. The resulting isosceles triangle can point in any of eight directions, and is denoted by $2+3+3$ in the table below.

Continuing this counting for one- four-, five-, six-, seven-, and eight-cornered figures, we obtain all 144 possible shapes for Manhattan polygons:

# Corners	Combining Which Corners	# Shapes
1	8	1
2	4+4	4
3	2+3+3	8
4	2+2+2+2, 3+2+2, 3+3	38
5	2+2+2, 3+2	56
6	2+2, 3	28
7	2	8
8	none	1

Manhattan polygons are similar to objects that others have investigated. If we insisted that each edge contained no lattice points other than the vertices, we would have a canonical polygon [10]. If, instead, we relaxed the restriction on edge slopes, we would have a lattice polygon [8]. Manhattan polygons are most closely related to the study of jump systems, because the convex hull of a two-dimensional jump system is known to be precisely a Manhattan polygon, and all Manhattan polygons arise this way.

We now define unit vectors N, S, E, W as $e_2, -e_2, e_1, -e_1$, respectively. We define $NE = N + E, NW = N + W, SE = S + E, SW = S + W$. The N face of a polygon is defined as those points x in the polygon maximizing $N^T x$. Set M_N to be this maximum. Other faces and M_i are defined similarly. A Manhattan polygon has eight (not necessarily distinct) faces corresponding to the eight above-mentioned vectors. It also has eight (not necessarily distinct) corners, corresponding to the intersection of two adjacent faces. For convenience, they will be denoted as the $NNE, ENE, SSE, ESE, SSW, WSW, NNW, WNW$ corners.

The N halfplane corresponding to a polygon is defined as those points y in the plane such that $N^T y \leq M_N$. The intersection of the eight halfplanes is exactly the polygon itself. Contrapositively, each point outside the polygon must lie outside at least one halfplane.

A *Manhattan set* is a convex subset of the two-dimensional integer lattice whose convex hull is a Manhattan polygon. Similarly, each Manhattan polygon has an associated Manhattan set. It is known that all Manhattan sets are jump systems, although not all two-dimensional jump systems arise in this way.

We now consider three dimensional objects. While analogous three-dimensional polytopes and sets can be defined, we instead consider more general objects – *laminated Manhattan sets*. That is, we choose a preferred axis, from now on the Z axis. We consider a collection of Manhattan sets in planes parallel to the X-Y plane, each corresponding to an integer Z-value. These objects include all of the analogous three-dimensional Manhattan sets (which are all laminates of two-dimensional Manhattan sets) and other three-dimensional objects as well.

Our main result is a set of conditions under which a laminated Manhattan set is a jump system. We first present a necessary weaker result, together with machinery needed for the main result.

3. LAMINATED MANHATTAN SETS

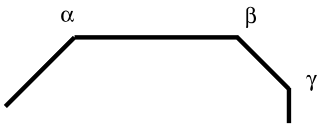
Our first theorem concerns the case of two layers only. The proof is fairly long, so we postpone it until the appendix.

THEOREM 3.1. *Let* $LMS = (MS_1 \times \{1\}) \cup (MS_2 \times \{2\}) \subseteq \mathbb{Z}^2 \times \{1,2\}$, *where* MS_1, MS_2 *are Manhattan sets. Then the following conditions are equivalent:*

1.LMS *is a jump system.*

2.For each corner κ' *of* MS_1*, the corresponding corner* κ'' *of* MS_2 *satisfies* $|\kappa' - \kappa''| \leq 1$.

Observe that the eight restrictions imposed by the second condition are not independent, because of the properties of the Manhattan sets. To illustrate this, we consider the NNW, NNE, and ENE corners, denoted by $\alpha', \alpha'', \beta', \beta'', \gamma', \gamma''$ respectively.



If $\beta' = \beta''$, then $\gamma' = \gamma''$ and $\alpha' = \alpha'' + \{-1,0,1\}W$. If $\beta' = \beta'' + N$, then $\alpha' = \alpha'' + N$ and $\gamma' = \gamma'' + \{N, E\}$. If $\beta' = \beta'' + E$, then $\alpha' = \alpha'' + \{-1,0,1\}W$ and $\gamma' = \gamma'' + \{N, E\}$.

The condition (2) implies the following weaker condition (2').

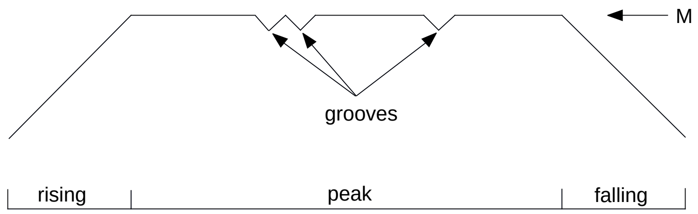
2'. For each of the eight faces f, we have $|M'_f - M''_f| \leq 1$, where M'_f, M''_f denotes the maximum of $f^T x$ on MS_1, MS_2, respectively.

We will extend this result from two layers to arbitrarily many layers. But first we need the following definition.

DEFINITION 3.1. A sequence of integers $a_1, a_2, \ldots, a_n$ is called *grooved tight unimodal* (GTU) if for all possible i, the following hold:

1. $|a_i - a_{i+1}| \leq 1$ (tight)
2. If any of $a_{i+1}, a_{i+2}, \ldots, a_n$ are greater than a_i, then $a_{i+1} = a_i + 1$.
3. If any of $a_1, a_2, \ldots, a_{i-1}$ are greater than a_i, then $a_{i-1} = a_i + 1$.

This definition uniquely decomposes the sequence into three subsequences. First is a (possibly empty) rising subsequence, where each term is one more than the last. Then is a peak subsequence, where all terms are either equal to the maximum of the sequence (M) or one less. Those that are one less are called grooves and must be both immediately preceded and followed by M terms. Finally comes a (possibly empty) falling subsequence, where each term is one less than the previous. A picture of this structure follows.

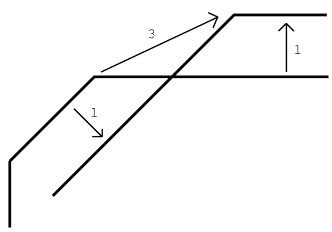


With this understanding we are ready to state our main result.

THEOREM 3.2. *Let $LMS = (MS_1 \times \{1\}) \cup (MS_2 \times \{2\}) \cup \cdots \cup (MS_n \times \{n\}) \subseteq \mathbb{Z}^2 \times \{1, 2, \ldots, n\}$, where $MS_1, MS_2, \ldots, MS_n$ are Manhattan sets. Then LMS is a jump system if and only if the following conditions hold:*

1.Each pair of consecutive layers, taken by themselves, forms a jump system.

2.For each of the eight faces f, we have $\{M_f^i\}$ a GTU sequence, where M_f^i denotes the maximum of $f^T x$ on MS_i.



Observe that the eight GTU restrictions are not independent. For example, if M_N increases and M_{NW} decreases from one layer to the next, then the NNW corner moves 3 units away, which is impossible by the first condition. In fact, if M_N (similarly M_E, M_S, M_W) changes, then M_{NW} and M_{NE} must change the same way. If M_{NW} (similarly M_{NE}, M_{SE}, M_{SW}) changes, then M_N and M_W cannot change in the opposite direction.

Proof. First, we assume that LMS is a jump system. The first condition must be true since the intersection of a jump system with a box is again a jump system. Suppose that one of the faces f fails the GTU criterion. It cannot fail the first part of the definition, because of Theorem 3.1. By renumbering the Manhattan sets if necessary, we may assume without loss that it fails the second part of the definition. Further, by renumbering we may even impose without loss that $M_f^1 \geq M_f^2$, but that $M_f^1 < M_f^m$ for some $m > 2$. Consider all $x \in (MS_1, 1)$ with $f^T x = M_f^1$, and all $y \in (MS_m, m)$ with $f^T y = M_f^m$. From these, choose x, y with $|x-y|$ minimal. Let z be any step from x toward y that is in the first layer and has $f^T z > f^T x = M_f^1$. Such a step must exist since $f^T z \leq f^T y = M_f^m$. We note that z is not in $(MS_1, 1)$, and that any step z' from z toward y that is in $(MS_1, 1)$ would violate the assumed minimality of $|x - y|$. The only possible other step would be if z' were in $(MS_2, 2)$, but we have $f^T z' = f^T z > M_F^1 \geq M_f^2$, so $z' \notin (MS_2, 2)$. This is a contradiction, so the GTU criterion must hold.

We now suppose that conditions (1) and (2) hold, but that LMS is not a jump system. In that case, there must be some $x \xrightarrow{y} z$, with $x, y \in LMS$, but $z \notin LMS$ and further no second step will be in LMS. Observe that x, y cannot be in the same or adjacent layers, by condition (1). By renumbering the Manhattan sets if necessary, we assume without loss that x is in layer 1, and that y is in layer m for some $m > 2$.

By swapping coordinates and reflection, if necessary, we may assume without loss that either $z = x + E$ (any step in the same layer as x) or $z = x + e_3$. We shall treat these cases separately.

First, we consider the case of $z = x + E$. Suppose that z is outside the E halfplane of MS_1. We will show that $z' = z + e_3$ is in MS_2. Because $x_1 < y_1$, we must have $M_E^2 > M_E^1$, hence z' is in the E halfplane. This also implies that $M_{NE}^2 > M_{NE}^1$ and $M_{SE}^2 > M_{SE}^1$; hence since x was in those halfplanes of MS_1, z' must be in those halfplanes of MS_2. In turn, we also conclude that $M_N^2 \geq M_N^1$ and $M_S^2 \geq M_S^1$, and again since x was in those halfplanes in MS_1, we must have z' in those halfplanes of MS_2. Finally, since the M_{SW}, M_{NW}, M_W are GTU sequences, they can decrease by at most one between MS_1 and MS_2. Therefore, since x was in those halfplanes in MS_1, again z' must be in those halfplanes in MS_2. Hence, z' is in all eight halfplanes of MS_2 and hence in MS_2.

Now, if z is not outside the E halfplane of MS_1, we may assume without loss (by reflection), that z is outside the NE halfplane rather than the SE halfplane. Observe that since z is not outside the E halfplane, we must have $z' = z + S \in MS_1$. If $y_2 < z_2$, then z' is a step from z in the direction of y, and x, y, z don't contradict the jump system axiom. Otherwise, $y_2 \geq z_2 = x_2$, and $y_1 > x_1$. Therefore, $M_{NE}^m > M_{NE}^1$, and hence $M_{NE}^2 > M_{NE}^1$. We will now show that $z'' = z + e_3$ is in MS_2. Because z was just outside the NE halfplane of MS_1, and $M_{NE}^2 > M_{NE}^1$, z'' is inside the NE halfplane of MS_2. Because the M_{NE} increased, M_N and M_E cannot decrease. Since z was inside both N and E halfplanes in MS_1, z'' is inside both in MS_2. Because of x and z', all the other five halfplanes had z at least one unit deep within the border. Since they can only change by one unit, z'' must still be inside these halfplanes in MS_2. Hence z'' is in all eight halfplanes in MS_2 and hence in MS_2.

Now, we consider the case that $z = x + e_3$. Consider the four points $z+E, z+S, z+W, z+N$. We claim that none are in MS_2. Suppose one of them, z', is in MS_2. If z' is in the direction of y, then x, y, z satisfies the jump system axiom, contrary to hypothesis. If z' is not in the direction of y, then z is a step from z' toward y, and we have a violation z', y, z of the jump system axiom where the step is in the same layer. This case has been handled previously. Hence all four points are not in MS_2, together with z. They form a Manhattan set, disjoint from MS_2, and hence are all outside at least one of the halfplanes (say M_A^2) of MS_2, since both sets are convex. Because M_A forms a GTU sequence, $|M_A^1 - M_A^2| \leq 1$, and hence x is outside M_A^1, contradicting the hypothesis that $x \in MS_1$. ∎

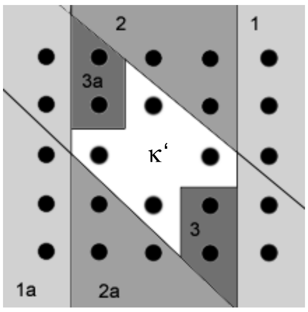
4. APPENDIX

We now give the promised proof of Theorem 3.1. We repeat the theorem here for convenience.

Theorem 3.1. *Let $LMS = (MS_1 \times \{1\}) \cup (MS_2 \times \{2\}) \subseteq \mathbb{Z}^2 \times \{1,2\}$, where MS_1, MS_2 are Manhattan sets. Then the following are equivalent:*

1.LMS is a jump system.
2.For each corner κ' of MS_1, the corresponding corner κ'' of MS_2 satisfies $|\kappa' - \kappa''| \leq 1$.

Proof ($1 \to 2$ in Theorem 3.1).
By symmetry, we assume without loss that κ', κ'' are the ENE corners of MS_1, MS_2 respectively, and that $|\kappa' - \kappa''| \geq 2$. The proof proceeds in three cases, depending on the relationship of κ'' to κ', as depicted below.



Case 1 is when $(\kappa'')_1 \geq (\kappa')_1 + 2$. Case 2 is when $(\kappa'')_1 + (\kappa'')_2 \geq (\kappa')_1 + (\kappa')_2 + 2$. Case 3 is very specific, it is when $(\kappa'')_1 = (\kappa')_1 + 1$ and $(\kappa')_2 - 2 \leq (\kappa'')_2 \leq (\kappa')_2 - 1$. Without loss of generality we may ignore the other cases – by interchanging the roles of κ', κ'', the picture is rotated 180° about the origin and case 1a is mapped to case 1, etc.

(Case 1) Consider the step $(\kappa'', 2) \xrightarrow{(\kappa',1)} (\kappa'', 1)$. Both $(\kappa'', 2)$ and $(\kappa', 1)$ are in LMS. However, $(\kappa'', 1)$ is not, since $(\kappa', 1)$ is on the east face of that layer, and $(\kappa')_1 \leq (\kappa'')_1 - 2$. Further, any possible step from $(\kappa'', 1)$ toward $(\kappa', 1)$ must stay in that layer, and cannot change the first coordinate by more than one. Therefore no second step can be in LMS, hence LMS violates the jump system axiom.

(Case 2) Similarly, consider the step $(\kappa'', 2) \xrightarrow{(\kappa',1)} (\kappa'', 1)$. Both $(\kappa'', 2)$ and $(\kappa', 1)$ are in LMS. However, $(\kappa'', 1)$ is not, since $(\kappa', 1)$ is on the northeast face of that layer, and $(\kappa')_1 + (\kappa')_2 \leq (\kappa'')_1 + (\kappa'')_2 - 2$. Further, any possible step from $(\kappa'', 1)$ toward $(\kappa', 1)$ must stay in that layer, and cannot change the sum of the coordinates by more than one. Therefore no second step can be in LMS, hence LMS violates the jump system axiom.

(Case 3) Consider the step $(\kappa', 1) \xrightarrow{(\kappa'',2)} (\kappa' + E, 1)$. This is not in LMS, since $(\kappa', 1)$ was in the east face of MS_1. Because the first coordinate of $(\kappa' + E, 1)$ is the same as that of $(\kappa'', 2)$, the only possible second steps are to $(\kappa' + E, 2)$ and $(\kappa' + E + S, 1)$. The first is not in LMS, because $(\kappa' + E)_1 + (\kappa' + E)_2 = (\kappa')_1 + (\kappa')_2 + 1 \geq (\kappa'')_1 + (\kappa'')_2 + 1$, but κ'' is on the northeast face of MS_2. The second is not in LMS, because $(\kappa' + E + S)_1 = (\kappa')_1 + 1$, and κ' is in the east face of MS_1. Hence, no second step can be in LMS, so LMS violates the jump system axiom. ▮

Proof $(2 \to 1$ in Theorem 3.1). Because LMS is not a jump system, there must be a violation of the jump system axiom. This must involve points from both layers, since otherwise a MS would violate the jump system axiom. By relabeling and reflection if necessary, we assume without loss of generality that the violation is $(x, 1) \xrightarrow{(y,2)} \bar{z}$. We have $(x, 1), (y, 2) \in LMS$, but $\bar{z} \notin LMS$. Furthermore, all possible steps $\bar{z} \xrightarrow{(y,2)} \bar{z}'$ have $\bar{z}' \notin LMS$. Now $\bar{z}$ can be one of five vectors. One of these is $\bar{z} = (x, 2)$. By reflection and coordinate-swapping if necessary, we can without loss of generality reduce all four other possibilities to $\bar{z} = (x+E, 1)$.

(Case 1) $\bar{z} = (x, 2)$.

Because $(x, 2) \notin LMS$, there must be at least one face of MS_2 that x is beyond. By reflection and coordinate-swapping if necessary, we can reduce these eight cases to two. Either x is outside the E halfplane of MS_2, or x is outside the NE halfplane of MS_2.

First suppose that x is outside the E halfplane of MS_2. Let κ', κ'' be the ENE corners of MS_1, MS_2, respectively. Because x is outside the E halfplane, we have $(x, 2) \xrightarrow{(y,2)} (x - E, 2)$. This point too is not in LMS. If this is still outside the E halfplane, then $(\kappa'')_1 \leq (x - E)_1 - 1 = x_1 - 2 \leq (\kappa')_1 - 2$, which violates our hypothesis. Otherwise, $x - E$ is not outside the E halfplane of MS_2, but still outside MS_2. Hence, without loss of generality, it is outside the NE halfplane of MS_2. We can get two equations from these facts: $(\kappa'')_1 = x_1 - 1 \leq (\kappa')_1 - 1$, and $(\kappa'')_2 \leq x_2 - 1 \leq (\kappa')_2 - 1$.

Putting them together, we get $|\kappa' - \kappa''| \geq 2$, which again violates our hypothesis.

Hence, we must have that x is outside the NE halfplane of MS_2, but in the N and E halfplanes of MS_2. We are assuming condition (2), and hence the weaker condition (2'). Therefore, both $\bar{z}+W$ and $\bar{z}+S$ are in the NE halfplane of MS_2. Further, they are both in MS_2, since $\bar{z}$ is in the N and E halfplanes of MS_2. However, one of them must be in the direction of y, violating our hypothesis that no second step is in LMS.

(Case 2) $\bar{z} = (x+E, 1)$.

First we suppose that $y_2 = x_2$. We have $\bar{z} \xrightarrow{(y,2)} (x+E, 2)$. Observe that $x+E$ is in the E, NE, SE, N, S halfplanes of MS_2. However, it cannot be in MS_2, so it must be outside some halfplane. By reflection, we may assume that it is either the W or NW halfplane. Let κ', κ'' be the WNW corners of MS_1, MS_2, respectively. If $x+E$ is outside the W halfplane, then observe that $(\kappa')_1 + 2 \leq x_1 + 2 = (x+E)_1 + 1 \leq (\kappa'')_1$. If $x+E$ is outside the NW halfplane, then observe that $-x_1 - 1 + x_2 \geq -\kappa''_1 + \kappa''_2 + 1$. But since x is in the NW halfplane of MS_1, we have $-x_1 + x_2 \leq -\kappa'_1 + \kappa'_2$. Combining, we have $\kappa''_1 - \kappa'_1 + \kappa'_2 - \kappa''_2 \geq 2$, which violates our hypothesis.

We henceforth assume, without loss of generality, that $y_2 < x_2$. Therefore, $x+E$ must be outside the E or SE halfplane of MS_1, since otherwise it would be outside the NE halfplane and $(x+E, 1) \xrightarrow{(y,2)} (x+E+S)$ would be in LMS and violate hypothesis.

Let κ', κ'' be the ESE corners of MS_1, MS_2 respectively. If $x+E$ is outside the SE halfplane of MS_1, then $(\kappa')_1 - (\kappa')_2 + 1 \leq (x+E)_1 - (x+E)_2 < y_1 - y_2 \leq (\kappa'')_1 - (\kappa'')_2$. In other words, $(\kappa')_1 - (\kappa')_2 + 2 \leq (\kappa'')_1 - (\kappa'')_2$, which violates our hypothesis.

Hence we must have $x+E$ outside the E halfplane of MS_1. Hence, $(\kappa')_1 = x_1$. However, since $(\kappa'')_1 \geq y_1 > x_1$, we must have $\kappa'' = \kappa' + E$. Now we set η', η'' to be the ENE corners of MS_1, MS_2, respectively. We know that $(\eta')_1 + 1 = (\eta'')_1$. However, $(\eta'')_2 < x_2$, since $x+E \notin MS_2$. But $(\eta')_2 \geq x_2$, since x is in the E face of MS_1. This violates our hypothesis. ∎

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