THE CATENARY AND TAME DEGREE IN FINITELY GENERATED COMMUTATIVE CANCELLATIVE MONOIDS

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Abstract. Problems involving chains of irreducible factorizations in atomic integral domains and monoids have been the focus of much recent literature. If S is a commutative cancellative atomic monoid, then the catenary degree of S (denoted $c(S)$) and the tame degree of S (denoted $t(S)$) are combinatorial invariants of S which describe the behavior of chains of factorizations. In this note, we describe methods to compute both $c(S)$ and $t(S)$ when M is a finitely generated commutative cancellative monoid.

1. Introduction

The study of combinatorial properties of non-unique factorizations in integral domains and monoids has become an active area of interest (see [9] and its references). Early work in this area focused on study of the *elasticity of factorization* which describes non-unique factorizations in a "coarse" sense (see for instance [2] where the first, second and fifth authors of the current paper construct an algorithm to compute the elasticity of a Krull monoid with finite divisor class group). Recently, the study of more precise invariants associated to non-unique factorizations has become popular (see for instance the papers [6], [7], [8], [5] and [11]). The two principal such invariants are known as the *catenary degree* and the *tame degree*. A summary of the up to date status of research concerning these constants can be found in [9, Chapters 6.4 and 6.5], but needless to say, exact computations of these constants (especially in the case of the tame degree) are not abundant. In Section 3 of this paper, we will describe two methods to compute the catenary degree of a finitely generated commutative cancellative monoid S. These methods are based on the computation of a minimal presentation of S, and we review these computations in Section 2. The material on presentations draws heavily on results from [3] and [14]. Our computations with the catenary degree will lead to a similar method in Section 4 to compute the tame degree of S. We close Section 4 with several examples illustrating the functionality of our results. All programming involving our algorithms was implemented in *GAP* [17]. While consideration of the finitely generated case my seem to be a strong restriction, many classes of monoids, such as numerical monoids (see [1]), block monoids over finite abelian groups (see [16]),

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and Diophantine monoids (see [4]) can be studied under this assumption. Since block monoids fall on this list, our methods are applicable to any Krull monoid whose divisor class group contains finitely many prime divisors (see [9, Theorem 2.5.8]).

We open with some notation and definitions. S will always denote a commutative cancellative monoid with set of units $S^\times$. Since units are not relevant to the study of the factorization properties of S, by passing (if necessary) to the quotient monoid $S/S^\times$, we can always assume that S is *reduced* (i.e., $|S^\times| = 1$). We assume throughout that $\{n_1, \ldots, n_p\}$ is the minimal system of generators of S. The map

$$\varphi : \mathbb{N}^p \to S,\ \varphi(a_1, \ldots, a_p) = a_1 n_1 + \cdots + a_p n_p$$

is a monoid homomorphism, known as the *factorization homomorphism* of S. Let σ be its kernel congruence, that is, $a\sigma b$ if and only if $\varphi(a) = \varphi(b)$. Then S is isomorphic to $\mathbb{N}^p/\sigma$ and for $n \in S$, the set $\varphi^{-1}(n)$ is the set of *factorizations* of n. Under our hypothesis, this set is always finite (see [14]).

If $(a_1, \ldots, a_p) \in \varphi^{-1}(n)$, then of course $n = a_1 n_1 + \cdots + a_p n_p$ and the *length* of the factorization $a = (a_1, \ldots, a_p)$ is $|a| = a_1 + \cdots + a_p$. For $z = (z_1, \ldots, z_p), z' = (z'_1, \ldots, z'_p) \in \mathbb{N}^p$ write

$$\gcd(z, z') = (\min\{z_1, z'_1\}, \ldots, \min\{z_p, z'_p\})$$

and

$$\frac{z}{z'} = z - z'.$$

Define

$$\mathrm{d}(z, z') = \max\left\{\left|\frac{z}{\gcd(z, z')}\right|, \left|\frac{z'}{\gcd(z, z')}\right|\right\},$$

to be the *distance* between z and z'. The basic properties of the distance function can be found in [9, Proposition 1.2.5]. The *support* of $z \in \mathbb{N}^p$ is defined as usual by

$$\mathrm{supp}(z) = \{i \in \{1, \ldots, p\} \mid z_i \neq 0\}.$$

Given $n \in S$ and $z, z' \in \varphi^{-1}(n)$, then an *N-chain of factorizations* from z to z' is a sequence $z_0, \ldots, z_k \in \varphi^{-1}(s)$ such that $z_0 = z$, $z_k = z'$ and $\mathrm{d}(z_i, z_{i+1}) \leq N$ for all i. The *catenary degree* of n, $\mathrm{c}(n)$, is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for any two factorizations $z, z' \in \varphi^{-1}(n)$, there is an N-chain from z to z'. The catenary degree of S, denoted by $\mathrm{c}(S)$, is defined by

$$\mathrm{c}(S) = \sup\{\mathrm{c}(n) \mid n \in S\}.$$

The *tame degree* $\mathrm{t}_S(S', X)$ of $S' \subseteq S$ and $X \subseteq \mathbb{N}^p$ is the minimum of all $N \in \mathbb{N} \cup \{\infty\}$ such that for all $s \in S'$, $z \in \varphi^{-1}(s)$ and $x \in X$ such that $s - \varphi(x) \in S$, there exists $z' \in \varphi^{-1}(s)$ such that $x \leq z'$ and $\mathrm{d}(z, z') \leq N$. For ease of notation, we write $\mathrm{t}_S(S', x)$ instead of $\mathrm{t}_S(S', \{x\})$, and $\mathrm{t}(S, X)$ instead of $\mathrm{t}_S(S, X)$. The monoid S is said to be *locally tame* if $\mathrm{t}(S, n_i)$ is finite for all $i \in \{1, \ldots, p\}$, and *tame* if $\mathrm{t}(S) = \mathrm{t}(S, \{n_1, \ldots, n_p\}) < \infty$. Clearly, as S is finitely generated, both concepts are equivalent. This is not the case in general for non-finitely generated monoids (see [9, Theorem 1.6.7].

2. Presentations of finitely generated monoids

By Redei's theorem (see [12]), every finitely generated commutative monoid is finitely presented. That is, there exists $\rho = \{(a_1, b_1), \ldots, (a_t, b_t)\} \subset \mathbb{N}^p \times \mathbb{N}^p$ such that the kernel congruence of φ is the least congruence containing ρ. In [14] an algorithm for finding a minimal presentation for S is given, that is, a set generating ρ such that none of its proper subsets generates σ. We review briefly this procedure.

For every $n \in S$, define G_n to be the graph with vertices

$$V_n = \{n_i \mid n - n_i \in S\}$$

and whose edges are

$$E_n = \{n_i n_j \mid n - (n_i + n_j) \in S\}.$$

Given $s \in S$ and $a, b \in \varphi^{-1}(s)$, we write $a\mathcal{R}b$ if there exists a chain $a_1, \ldots, a_k \in \varphi^{-1}(s)$ such that

- $a_1 = a$, $a_k = b$,
- for all $i \in \{1, \ldots, k-1\}$, $\operatorname{supp}(a_i) \cap \operatorname{supp}(a_{i+1}) \neq \emptyset$.

It can be shown that the number of connected components of G_n coincides with the number of $\mathcal{R}$-classes of $\varphi^{-1}(n)$. For every $n \in S$, define ρ_n in the following way.

- If G_n is connected, then set $\rho_n = \emptyset$.
- If G_n is not connected and $\mathcal{R}_1, \ldots, \mathcal{R}_k$ are the different $\mathcal{R}$-classes of $\varphi^{-1}(n)$, then choose $z_i \in \mathcal{R}_i$ for all $i \in \{1, \ldots, k\}$ and set $\rho_n = \{(z_1, z_2), \ldots, (z_1, z_k)\}$.

Then $\rho = \bigcup_{n \in S} \rho_n$ is a minimal presentation of S (moreover, in this way you can construct all minimal presentations for S). There are finitely many elements n in S for which G_n is not connected.

There is another approach for the construction of a minimal presentation for S, which is related to the set of nonnegative solutions of a system of linear Diophantine equations. Under the standing hypothesis, S can be embedded in $\mathbb{Z}^k \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ for some positive integers $k, d_1, \ldots, d_r$ with $d_1 | d_2 | \cdots | d_r$ (see for instance [15]). So $n_1, \ldots, n_p$ can be viewed as elements in $\mathbb{Z}^k \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$. Let M be the subgroup of $\mathbb{Z}^p$ whose defining equations are

$$n_1 x_1 + \cdots + n_p x_p = 0$$

(where this is understood to be $k + r$ equations and zero is the zero of $\mathbb{Z}^k \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$). It follows that (see for instance [15]) $\sigma = \operatorname{Ker}(\varphi) = \sim_M$, where

$$\sim_M = \{(a, b) \in \mathbb{N}^p \times \mathbb{N}^p \mid a - b \in M\}.$$

The set of irreducibles of $\sim_M$, $\mathcal{I}(\sim_M)$, is the set of nontrivial minimal elements with respect to the usual partial order $\leq$ on $\mathbb{N}^p \times \mathbb{N}^p$. From [15, Chapter 8], it can be shown that $((x_1, \ldots, x_p), (y_1, \ldots, y_p)) \in \mathcal{I}(\sim_M)$ if and only if $(x_1, \ldots, x_p, y_1, \ldots, y_p)$ is a minimal nontrivial nonnegative solution to the system of equations

$$n_1 x_1 + \cdots + n_p x_p - n_1 y_1 - \cdots - n_p y_p = 0. \tag{1}$$

The kernel congruence of φ is generated by $\mathcal{I}(\sim_M)$ as a monoid, and thus as a congruence. This means that if $\varphi(z) = \varphi(z')$, then $(z, z') = \sum_{i=1}^{k} (a_i, b_i)$ with $(a_i, b_i) \in \mathcal{I}(\sim_M)$ for all i.

Remark 1. Usually if ρ is a minimal presentation, then the set $\mathcal{I}(\sim_M)$ is larger and contains much more information. The reader can check this in the examples given in the next sections. However, it is easy to prove that $\mathcal{I}(\sim_M)$ contains all possible minimal presentations of S. In order to see this, define the following relation on S: for $a, b \in S$, $a \le_S b$ if $a + c = b$ for some $c \in S$. We use $a <_S b$ to denote $a \le_S b$ and $a \neq b$. The reader can check that $\le_S$ is reflexive, transitive and antisymmetric since S is a cancellative and reduced monoid. Let (a, b) be an element in a minimal presentation ρ of S. Assume that $(a, b) \notin \mathcal{I}(\sim_M)$. As the set of irreducibles generates σ as a monoid, this in particular means that, there must be an element $(a', b') \in \mathcal{I}(\sim_M)$ such that $(a', b') < (a, b)$. Thus $\varphi(a') = \varphi(b') <_S \varphi(a) = \varphi(b)$. As ρ generates σ as a congruence, there exists a sequence $u_1, \ldots, u_t \in \mathbb{N}^p$ such that $a' = u_1 \sigma u_2 \sigma \cdots \sigma u_{t-1} \sigma u_t = b'$ and $(u_i, u_{i+1}) = (a_i + v_i, b_i + v_i)$ with $(a_i, b_i) \in \rho \cup \rho^{-1}$ for all $i \in \{1, \ldots, t-1\}$ where $\rho^{-1} = \{(b, a) \mid (a, b) \in \rho\}$. But then $\varphi(a') = \varphi(a_i) + \varphi(u_i) = \varphi(b_i) + \varphi(u_i)$. Hence $\varphi(a_i) <_S \varphi(a') <_S \varphi(s)$, which leads to $(a_i, b_i) \neq (a, b)$ for any i. This would imply that (a, b) can be obtained from the rest of elements in ρ, contradicting the minimality of ρ.

3. The catenary degree in finitely generated commutative cancellative monoids

Recall (as illustrated above) that if ρ is a minimal presentation for S (that is, a minimal system of generators of σ as a congruence), then whenever $z \sigma z'$, there exists $z_0, \ldots, z_k \in \mathbb{N}^p$ in such a way that

$$z = z_0 \sigma z_1 \sigma \cdots \sigma z_{k-1} \sigma z_k = z'$$

and $(z_i, z_{i+1}) = (a_i + u_i, b_i + u_i)$ for some $u_i \in \mathbb{N}^p$ and $(a_i, b_i) \in \rho \cup \rho^{-1}$. Notice that if $(a, b) \in \rho \cup \rho^{-1}$, then the set $\mathrm{supp}(a) \cap \mathrm{supp}(b)$ is empty, and thus $\gcd(a, b) = 0$. This in particular implies that $\mathrm{d}(a, b) = \max\{|a|, |b|\}$. Observe also that $\mathrm{d}(a + u, b + u) = \mathrm{d}(a, b)$. Thus in the above chain $z_0, \ldots, z_p$ the distance between two adjacent elements is bounded by $\max\{|a| \mid (a, b) \in \rho \cup \rho^{-1}$ for some $b \in \mathbb{N}^p\}$.

From the above remark, we obtain the following.

Proposition 2. *Let* $|\rho| = \max\{|a| \mid (a, b) \in \rho \cup \rho^{-1}$ *for some* $b \in \mathbb{N}^p\}$. *Then*

$$\mathrm{c}(S) \le |\rho|.$$

By using this together with Lambert's bound [10] for a single linear homogeneous Diophantine equation, we obtain this consequence for numerical monoids.

Corollary 3. *Assume that* $n_1, \ldots, n_p$ *are positive integers. Then*

$$\mathrm{c}(\langle n_1, \ldots, n_p \rangle) \le \max\{n_1, \ldots, n_p\}.$$

Proof. Let $S = \langle n_1, \ldots, n_p \rangle$ and let ρ be a minimal presentation of S. As we have seen above, if $(a, b) \in \rho$, then $(a, b) \in \mathcal{I}(\sim_M)$, or in other words, if $(a, b) = ((a_1, \ldots, a_p), (b_1, \ldots, b_p))$, then $(a_1, \ldots, a_p, b_1, \ldots, b_p)$ is a minimal solution to the equation

$$n_1 x_1 + \cdots + n_p x_p - n_1 x_1 - \cdots - n_p x_p = 0.$$

By Lambert's bound (see [10]), we have that $a_1 + \ldots + a_p \le \max\{n_1, \ldots, n_p\}$ and $b_1 + \ldots + b_p \le \max\{n_1, \ldots, n_p\}$. Thus $|\rho| \le \max\{n_1, \ldots, n_p\}$. □

Let $n \in S$ be such that G_n is not connected and let $\mathcal{R}_1^n, \ldots, \mathcal{R}_{k_n}^n$ be its different $\mathcal{R}$-classes. Set $\mu(n) = \max\{r_1^n, \ldots, r_{k_n}^n\}$, where $r_i^n = \min\{|z| : z \in \mathcal{R}_i^n\}$. Define

$$\mu(S) = \max\{\mu(n) \mid n \in S \text{ and } \mathrm{G}_n \text{ not connected}\}.$$

Theorem 4. *Let S be a finitely generated commutative cancellative reduced monoid. Then*

$$\mathrm{c}(S) = \mu(S).$$

Proof. Construct ρ in the following way. For every $n \in S$ such that G_n is not connected, choose $(z_1^n, \ldots, z_{k_n}^n) \in \mathcal{R}_1^n \times \cdots \times \mathcal{R}_{k_n}^n$ such that $|z_i^n| = r_i^n$ for $i \in \{1, \ldots, k_n\}$. Take $\rho_n = \{(z_1^n, z_2^n), (z_1^n, z_3^n), \ldots, (z_1^n, z_{k_n}^n)\}$. If G_n is connected, then set $\rho_n = \emptyset$. Then, as pointed out above, $\rho = \bigcup_{n \in S} \rho_n$ is a minimal presentation for S. In view of Proposition 2, we deduce that $\mathrm{c}(S) \leq \mu(S)$.

Let $n \in S$ be such that $\mu(S) = \mu(n)$, and assume without loss of generality that $\mu(n) = |z_1^n|$. If $\mathrm{c}(S) < \mu(S)$, then $\mathrm{c}(n) < |z_1^n|$, or in other words, factorizations of n can be joined by c-chains for some $c < |z_1^n|$. Let $z = z_1^n$ and $z' = z_2^n$. Since z and z' are different factorizations of n, there must be a chain $z_1, \ldots, z_k$ of factorizations of n with $z_1 = z$, $z_k = z'$ and $\mathrm{d}(z_i, z_{i+1}) \leq c$. As z and z' are in different $\mathcal{R}$-classes, there exists $i \in \{1, \ldots, k\}$ such that $z = z_1, \ldots, z_i \in \mathcal{R}_1^n$ and $z_{i+1} \notin \mathcal{R}_1^n$. From the definition of $\mathcal{R}$-class, this in particular implies that $\mathrm{supp}(z_i) \cap \mathrm{supp}(z_{i+1})$ is empty. Hence $\mathrm{d}(z_i, z_{i+1}) = \max\{|z_i|, |z_{i+1}|\}$. As $z_i \in \mathcal{R}_1^n$ and $|z_1^n| = r_1^n = \min\{|z| : z \in \mathcal{R}_1^n\}$, we get that $|z_1^n| \leq |z_i|$. But then we obtain $|z_1^n| \leq \max\{|z_i|, |z_{i+1}|\} = \mathrm{d}(z_i, z_{i+1}) \leq c$, contradicting that $c < |z_1^n|$. □

Algorithms exist for computing minimal presentations, and also for computing all the expressions of an element in a finitely generated commutative cancellative reduced monoid. Thus, we can effectively compute $\mu(S)$ and in turn effectively compute $\mathrm{c}(S)$.

Example 5. We begin with a calculation involving a numerical monoid. Let $S = \langle n_1, \ldots, n_t \rangle$ with $t \geq 2$, $n_1, \ldots, n_t \in \mathbb{N}$, $1 < n_1 < \cdots < n_t$ and $\gcd(n_1, \ldots, n_t) = 1$. If $g(S)$ represents the Frobenius number of S, then from [9, Example 3.1.6] we have

$$c(S) \leq t(S) \leq \frac{g(S) + n_t}{n_1} + 1$$

and $c(S) = t(S) = n_2$ when $t = 2$. We consider the case $t = 3$ and let $S = \langle 3, 5, 7 \rangle$. The elements $n \in S$ such that G_n is not connected are 10, 12 and 14. And the different factorizations for each of these elements are

10: $(0, 2, 0)$ and $(1, 0, 1)$,
12: $(0, 1, 1)$ and $(4, 0, 0)$,
14: $(0, 0, 2)$ and $(3, 1, 0)$.

A minimal presentation for S is

$$\{((0, 2, 0), (1, 0, 1)), ((0, 1, 1), (4, 0, 0)), ((0, 0, 2), (3, 1, 0))\}.$$

Any two different factorizations are in different $\mathcal{R}$-classes. Then the catenary degree is the maximum of the lengths of these factorizations. Hence, $\mathrm{c}(S) = 4$. Observe that the bound given in Corollary 3 is far from sharp. □

We can rewrite this last theorem with respect to the irreducible elements of the associated linear Diophantine equation. If z and z' are different factorizations of the same element $n \in S$, then as mentioned above, $(z, z') = \sum_{i=1}^{k}(a_i, b_i)$ for some $(a_i, b_i) \in \mathcal{I}(\sim_M)$. We can construct the following chain of factorizations

$$z = z_0 \;\sigma\; z_1 = (z - a_1) + b_1 \;\sigma\; z_2 = (z - (a_1 + a_2)) + (b_1 + b_2) \;\sigma \cdots$$
$$\cdots \sigma\; z_k = (a - (a_1 + \cdots + a_k)) + (b_1 + \cdots + b_k) = z'.$$

The distance between z_i and z_{i+1} is $\mathrm{d}(z_i, z_{i+1}) = \mathrm{d}(a_{i+1} + (a - (a_1 + \cdots + a_i + a_{i+1}) + (b_1 + \cdots + b_i)), b_{i+1} + (a - (a_1 + \cdots + a_i + a_{i+1}) + (b_1 + \cdots + b_i))) = \mathrm{d}(a_{i+1}, b_{i+1})$. Since (a_{i+1}, b_{i+1}) is an irreducible solution of (1), then either $\mathrm{supp}(a_{i+1}) \cap \mathrm{supp}(b_{i+1})$ is empty or $a_i = b_i = e_j$ for some j. This implies that either $\mathrm{d}(z_i, z_{i+1}) = \max\{|a_{i+1}|, |b_{i+1}|\}$ or $\mathrm{d}(z_i, z_{i+1}) = 0$. In this way, one easily deduces the following result, similar to Proposition 2. Observe that if (a, b) belongs to $\mathcal{I}(\sim_M)$, then so does (b, a).

Proposition 6. *Let $|\mathcal{I}(\sim_M)| = \max\{|a| \mid (a, b) \in \mathcal{I}(\sim_M)$ for some $b \in \mathbb{N}^p\}$. Then*

$$\mathrm{c}(S) \le |\mathcal{I}(\sim_M)|.$$

Example 7. Let us return to the above example with $S = \langle 3, 5, 7\rangle$. If we compute the set of minimal solutions of the equation

$$3x_1 + 5x_2 + 7x_3 - 3y_1 - 5y_2 - 7y_3 = 0,$$

one obtains the following

$$\{((0,0,1),(0,0,1)),((0,0,2),(3,1,0)),((0,0,3),(2,3,0)),((0,0,3),(7,0,0)),$$
$$((0,0,4),(1,5,0)),((0,0,5),(0,7,0)),((0,1,0),(0,1,0)),((0,1,1),(4,0,0)),$$
$$((0,2,0),(1,0,1)),((0,3,0),(5,0,0)),((0,7,0),(0,0,5)),((1,0,0),(1,0,0)),$$
$$((1,0,1),(0,2,0)),((1,5,0),(0,0,4)),((2,3,0),(0,0,3)),((3,1,0),(0,0,2)),$$
$$((4,0,0),(0,1,1)),((5,0,0),(0,3,0)),((7,0,0),(0,0,3))\}$$

If we proceed analogously as we did for minimal presentations, we could determine which elements s in S have multiple factorizations in the above set. By inspection the elements 3, 5, 7, 10, 12, 14, 15, 21, 28 and 35 are those "involved" in the set $\mathcal{I}(\sim_M)$. Note for instance that the factorizations of 35 are $(0,0,5)$, $(0,7,0)$, $(1,5,1)$, $(2,3,2)$, $(3,1,3)$, $(5,4,0)$, $(6,2,1)$, $(7,0,2)$ and $(10,1,0)$ (but only $(0,0,5)$ and $(0,7,0)$ appear as part of an irreducible). Notice that the factorization of 35 with minimum length is $(0,0,5)$, and $|(0,0,5)| = 5 > 4 = \mathrm{c}(S)$. A 4-chain joining $(0,0,5)$ and $(0,7,0)$ is for instance $(0,0,5)\sigma(3,1,3)\sigma(1,5,1)\sigma(0,7,0)$.

So we cannot proceed as with minimal presentations. The main difference is that there is only one $\mathcal{R}$-class in the set of factorizations of 35, and when dealing with minimal presentations, every element in the semigroup involved in one of its minimal presentations has a set of factorizations with at least two $\mathcal{R}$-classes. □

One can S-grade the elements of $\mathcal{I}(\sim_M)$ in the following way. For $s \in S$, define $\mathcal{I}_s(\sim_M) = \{(a, b) \in \mathcal{I}(\sim_M) \mid \varphi(a) = s\}$. Then $\mathcal{I}(\sim_M) = \bigcup_{s \in S} \mathcal{I}_s(\sim_M)$. For every $s \in S$ such that $\mathcal{I}_s(\sim_M) \neq \emptyset$, let $\mathcal{R}_1^s, \ldots, \mathcal{R}_{k_s}^s$ be its different $\mathcal{R}$-classes of the set of factorizations of s. From each $\mathcal{R}$-class, choose an element a_i^s such that $|a_i^s|$ is minimum in its $\mathcal{R}$-class. Then among these, take a_s such that $|a_s| = \max\{|a_1^s|, \ldots, |a_{k_s}^s|\}$. Define

$$\nu(S) = \max\{|a_s| \mid s \in S, \mathcal{I}_s(\sim_M) \neq \emptyset, k_s > 1\}.$$

Theorem 8. *Let S be a finitely generated commutative cancellative reduced monoid. Then*

$$\mathrm{c}(S) = \nu(S).$$

Proof. In view of Theorem 4, it suffices to show that $\nu(S) = \mu(S)$. If we show that the elements $s \in S$ used to compute both $\nu(S)$ and $\mu(S)$ are the same, then we are done. That is to say, we must prove that $\{s \in S \mid \mathrm{G}_s \text{ not connected}\} = \{s \in S \mid \mathcal{I}_s(\sim_M) \neq \emptyset, k_s > 1\}$. Notice that if $s \in S$ and $k_s > 1$ (k_s as above stands for the number of $\mathcal{R}$-classes of factorizations of s), then G_s is not connected. Hence $\{s \in S \mid \mathcal{I}_s(\sim_M) \neq \emptyset, k_s > 1\} \subseteq \{s \in S \mid \mathrm{G}_s \text{ not connected}\}$. Now take $s \in S$ such that G_s is not connected. Then $k_s > 1$ and one can find two different factorizations of s, say a and b, such that (a, b) belongs to some minimal presentation of S. Then $(a, b) \in \mathcal{I}(\sim_M)$ (see Remark 1) and consequently $\mathcal{I}_s(\sim_M)$ is not empty. This proves the other inclusion. □

4. The tame degree and additional examples

Let S be a finitely generated commutative cancellative reduced monoid minimally generated by $\{n_1, \ldots, n_p\}$ and let φ be its factorization homomorphism. Let M be the subgroup of $\mathbb{Z}^p$ defined by $Ax = 0$, where A is the matrix whose columns are the generators of S. We already know that the kernel congruence of φ is $\sim_M$. Define, as above, $\mathcal{I}_s(\sim_M)$ as the set of irreducibles (a, b) of $\sim_M$ such that $\varphi(a) = s$ ($= \varphi(b)$). For $i \in \{1, \ldots, p\}$, set

$$F_s^i = \{a \in \varphi^{-1}(s) \mid i \in \mathrm{supp}(a)\}.$$

Given $Y \subseteq \mathbb{N}^p$ and $x \in \mathbb{N}^p$, define

$$\mathrm{d}(x, Y) = \min\{\mathrm{d}(x, y) \mid y \in Y\}.$$

Lemma 9. *With the notation introduced above, for every $i \in \{1, \ldots, p\}$,*

$$\mathrm{t}(S, n_i) = \max\{\mathrm{d}(a, F_{\varphi(a)}^i) \mid a \in \mathbb{N}^p, \varphi(a) - n_i \in S, \mathcal{I}_{\varphi(a)}(\sim_M) \neq \emptyset\}.$$

Proof. Let $t = \mathrm{t}(S, n_i)$ and $d = \max\{\mathrm{d}(a, F_{\varphi(a)}^i) \mid \varphi(a) - n_i \in S, \mathcal{I}_{\varphi(a)}(\sim_M) \neq \emptyset\}$. We first prove that $t \leq d$. Assume that $s \in S$ is such that $s - n_i \in S$. We have to show that there exists $z' \in F_s^i$ such that $\mathrm{d}(z, z') \leq d$. Let $z \in \varphi^{-1}(s)$. If $i \in \mathrm{supp}(z)$, then take $z = z'$. In this setting $\mathrm{d}(z, z') = 0 \leq d$. Now assume that $i \notin \mathrm{supp}(z)$. As $s - n_i \in S$, we can take $\bar{z} \in \varphi^{-1}(s - n_i)$. Then $\bar{z} + e_i \in \varphi^{-1}(s)$ and $i \in \mathrm{supp}(\bar{z} + e_i)$. Hence $(z, \bar{z} + e_i) \in \sim_M$. Thus, there exist $(a_1, b_1), \ldots, (a_t, b_t) \in \mathcal{I}(\sim_M)$ such that $(z, \bar{z} + e_i) = (a_1, b_1) + \cdots + (a_t, b_t)$. This implies that there exists $(a, b) \in \{(a_1, b_1), \ldots, (a_t, b_t)\}$ such that $(a, b) \leq (z, \bar{z} + e_i)$ and $i \in \mathrm{supp}(b)$ (observe that $a \leq z$ implies that $i \notin \mathrm{supp}(a)$). Then $(a, b) \in \mathcal{I}_{\varphi(a)}(\sim_M)$, $\varphi(a) - n_i \in S$ and $b \in F_{\varphi(a)}^i$. Take $b' \in F_{\varphi(a)}^i$ such that $\mathrm{d}(a, b') = \mathrm{d}(a, F_{\varphi(a)}^i)$. If we choose $z' = b' + z - a$, then $i \in \mathrm{supp}(z')$,

$z' \in \varphi^{-1}(s)$ and $\mathrm{d}(z, z') = \mathrm{d}(a + (z - a), b' + (z - a)) = \mathrm{d}(a, b') \leq d$. This proves that $t \leq d$.

For the other inequality, assume to the contrary that $t < d$. Let $a \in \mathbb{N}^p$ with $\varphi(a) - n_i \in S$ be such that $d = \mathrm{d}(a, F^i_{\varphi(a)})$, and let $b \in F^i_{\varphi(a)}$ be such that $d = \mathrm{d}(a, b)$. Then as t is the catenary degree of S, there must be an element $b' \in \varphi^{-1}(s)$ with $i \in \mathrm{supp}(b')$ and $\mathrm{d}(a, b') \leq t$. But this is impossible, since $d = \mathrm{d}(a, b) = \mathrm{d}(a, F^i_s) \leq \mathrm{d}(a, b') \leq t$. □

The following result now follows easily.

Proposition 10. *Let S be a finitely generated commutative cancellative reduced monoid minimally generated by $\{n_1, \ldots, n_p\}$. Then*

$$\mathrm{t}(S) = \max\{\mathrm{d}(a, F^i_{\varphi(a)}) \mid a \in \mathbb{N}^p, \varphi(a) - n_i \in S, \mathcal{I}_{\varphi(a)}(\sim_M) \neq \emptyset, \{1, \ldots, p\}\}.$$

We close with a series of examples which demonstrate the versatility of our methods.

Example 11. Let us revisit $S = \langle 3, 5, 7\rangle$. We already know from Example 7, that the elements of S for which $\mathcal{I}_s(\sim_M)$ is not empty are 3, 5, 7, 10, 14, 15, 21, 28 and 35. The first three are atoms, and for the rest their factorizations are

10: $(0, 2, 0)$ and $(1, 0, 1)$,
12: $(0, 1, 1)$ and $(4, 0, 0)$,
14: $(0, 0, 2)$ and $(3, 1, 0)$.
15: $(0, 3, 0)$, $(1, 1, 1)$ and $(5, 0, 0)$,
21: $(0, 0, 3)$, $(2, 3, 0)$, $(3, 1, 1)$ and $(7, 0, 0)$,
28: $(0, 0, 4)$, $(1, 5, 0)$, $(2, 3, 1)$, $(3, 1, 2)$, $(6, 2, 0)$ and $(7, 0, 1)$,
35: $(0, 0, 5)$, $(0, 7, 0)$, $(1, 5, 1)$, $(2, 3, 2)$, $(3, 1, 3)$, $(5, 4, 0)$, $(6, 2, 1)$, $(7, 0, 2)$ and $(10, 1, 0)$.

Thus we obtain the following table. For every element $s \in S$ with $\mathcal{I}_s(\sim_M) \neq \emptyset$ (first row) we found $\max\{\mathrm{d}(a, F^i_s) \mid a \in \varphi^{-1}(s)\}$ for $i \in \{1, 2, 3\}$ (second, third and fourth rows).

n_i \ S	10	12	14	15	21	28	35
3	2	4	4	2	4	4	4
5	2	4	4	4	4	4	4
7	2	4	4	4	4	2	2

In view of Proposition 10, $\mathrm{t}(S) = 4$. □

Example 12. If G is a finite abelian group, then we compute the catenary and tame degrees for the block monoid $\mathcal{B}(G, S)$ (see [9, Chapter 2.5]) where $S \subseteq G$ (if $S = G$, then we set $\mathcal{B}(G, G) = \mathcal{B}(G)$). From [9, Theorem 3.4.10], we have

$$c(\mathcal{B}(G, S)) \leq D(G, S) \text{ and } t(\mathcal{B}(G, S)) \leq 1 + \frac{D(G, S)(D(G, S) - 1)}{2}$$

where $D(G, S)$ represents the Davenport constant of G with respect to the subset S (see [9, Chapter 5.1]). Moreover $c(\mathcal{B}(\mathbb{Z}_n)) = n$ (see [9, Theorem 6.4.7]). Let $G =$

$\mathbb{Z}_5$ and $S = \{1,3\}$. The irreducible elements of $\mathcal{B}(\mathbb{Z}_5,\{1,3\})$ (written as vectors) are

$$n_1 = (0,5),\ n_2 = (1,3),\ n_3 = (2,1),\ \text{and } n_4 = (5,0).$$

The elements of $\mathcal{I}(\sim_{\mathbb{B}(\mathbb{Z}_5,S)})$ are as follows.

$$\begin{array}{lll}
\{((0,0,0,1),(0,0,0,1)), & ((0,0,1,0),(0,0,1,0)), & ((1,0,0,0),(1,0,0,0)),\\
((0,1,0,0),(0,1,0,0)), & ((0,0,3,0),(0,1,0,1)), & ((0,1,0,1),(0,0,3,0)),\\
((0,0,5,0),(1,0,0,2)), & ((1,0,0,2),(0,0,5,0)), & ((0,1,2,0),(1,0,0,1)),\\
((1,0,0,1),(0,1,2,0)), & ((0,2,0,0),(1,0,1,0)), & ((1,0,1,0),(0,2,0,0)),\\
((0,3,1,0),(2,0,0,1)), & ((2,0,0,1),(0,3,1,0)), & ((0,5,0,0),(3,0,0,1)),\\
((3,0,0,1),(0,5,0,0))\} & & .
\end{array}$$

The factorizations of the elements involved in these irreducibles are

(6,3): $(0,0,3,0)$ and $(0,1,0,1)$ representing two $\mathcal{R}$-classes.
(5,5): $(0,1,2,0)$ and $(1,0,0,1)$ representing two $\mathcal{R}$-classes.
(2,6): $(1,0,1,0)$ and $(0,2,0,0)$ representing two $\mathcal{R}$-classes.
(10,5): $(0,0,5,0)$ and $(1,0,0,2),(0,1,2,1)$ representing one $\mathcal{R}$-class.
(5,10); $(0,3,1,0)$, $(2,0,0,1)$ and $(1,1,2,0)$ representing one $\mathcal{R}$-class.
(5,15): $(0,5,0,0)$, $(3,0,0,1)$, $(1,3,1,0)$ and $(2,1,2,0)$ representing one $\mathcal{R}$-class.

Thus, $c(\mathcal{B}(\mathbb{Z}_5,\{1,3\})) = 3 < c(\mathcal{B}(\mathbb{Z}_5) = 5$. For the tame degree we obtain the following table.

n_i $\mathcal{B}$	(6,3)	(5,5)	(2,6)	(10,5)	(5,10)	(5,15)
n_1	-	3	2	3	2	2
n_2	3	3	2	3	3	3
n_3	3	3	2	3	3	3
n_4	3	3	-	3	3	5

Hence, $t(\mathcal{B}(\mathbb{Z}_5,\{1,3\})) = 5$. □

Example 13. By [9, Theorem 1.6.3], if S is an atomic monoid and $c(S) = 2$, then S is half-factorial (i.e., all irreducible factorizations of a nonunit element x have the same length). The converse of this statement is false, and as a counterexample the authors of [9] offer a half-factorial Dedekind domain D with $c(D) = \infty$. For each odd integer $n \geq 3$, we construct a half-factorial monoid S with $c(S) = n$. Let $n = 2k+1$ for some $k \geq 1$. Take $S = \langle (2,2+2n),(2,n),(2,0)\rangle$. Then S is isomorphic to $\mathbb{N}^3/\sim M$, with M given by the equations

$$2x + 2y + 2z = 0$$
$$(2+n)x + ny = 0.$$

A basis (as as subgroup of $\mathbb{Z}^3$) for M is $B = \{(n,-n-2,2)\}$ (here it is necessary that n is odd for otherwise both equations can be simplified and S is isomorphic to $\langle (1,1+n),(1,n/2),(1,0)\rangle$). Hence, by [3] the saturation of B is itself (since it has cardinality one) and thus the set $\mathcal{I}(\sim_M)$ is composed of the elements (e_i,e_i), $i \in \{1,2,3\}$, and $((n,0,2),(0,n+2,0))$ (and its symmetry). By the main result of [2], S is half-factorial. The only element (apart from the generators) involved in the irreducibles is $s = n(2, 2 + n) + 2(2, 0) = (n + 2)(2, n)$, which has only two different expressions (this follows since we have only two non-trivial irreducibles, say $((n, 0, 2), (0, n+2, 0))$ and its symmetry, $\{((n, 0, 2), (0, n+2, 0))\}$ and these form a minimal presentation for S; thus the associated graph G_s has only two connected components, whence s has only two $\mathcal{R}$-classes). Thus $C(S) = n + 2$. □

Example 14. We return to numerical monoids and show that the catenary degree may be strictly less than the tame degree in the case where S requires three generators. Set $S = \langle 10, 13, 15\rangle$. The defining equation for S is $M = 10x + 13y + 15z + 0$ and the irreducibles apart from (e_i, e_i), $i \in \{1, 2, 3\}$ in $\mathcal{I}(\sim_M)$, are

$$\begin{array}{lll}
\{((0,10,0),(13,0,0)), & ((0,5,1),(8,0,0)), & ((0,5,0),(5,0,1)), \\
((0,0,2),(3,0,0)), & ((0,5,0),(2,0,3)), & ((0,0,5),(1,5,0)), \\
((0,10,0),(1,0,8)), & ((0,0,13),(0,15,0)), & ((0,15,0),(0,0,13)), \\
((1,0,8),(0,10,0)), & ((1,5,0),(0,0,5)), & ((2,0,3),(0,5,0)), \\
((3,0,0),(0,0,2)), & ((5,0,1),(0,5,0)), & ((8,0,0),(0,5,1)), \\
((13,0,0),(0,10,0))\}. & &
\end{array}$$

The elements involved in these irreducibles are 30, 65, 75, 80, 130, 195. The only elements with more than two $\mathcal{R}$-classes are 30 and 65, and their different factorizations are

30: $(3, 0, 0)$, $(0, 0, 2)$
65: $(0, 5, 0)$, $(2, 0, 3)$.

Thus the catenary degree is 5. Factorizations for the rest of elements involved in the irreducibles are:

75: $(0, 0, 5)$, $(1, 5, 0)$, $(3, 0, 3)$, $(6, 0, 1)$,
80: $(0, 5, 1)$, $(2, 0, 4)$, $(5, 0, 2)$, $(8, 0, 0)$,
130: $(0, 10, 0)$, $(1, 0, 8)$, $(2, 5, 3)$, $(4, 0, 6)$, $(5, 5, 1)$, $(7, 0, 4)$, $(10, 0, 2)$, $(13, 0, 0)$,
195: $(0, 0, 13)$, $(0, 15, 0)$, $(1, 5, 8)$, $(2, 10, 3)$, $(3, 0, 11)$, $(4, 5, 6)$, $(5, 10, 1)$, $(6, 0, 9)$, $(7, 5, 4)$, $(9, 0, 7)$, $(10, 5, 2)$, $(12, 0, 5)$, $(13, 5, 0)$, $(15, 0, 3)$, $(18, 0, 1)$.

Then

- $t(S, n_1) = 6$ (reached in $d((0, 0, 5), \{(1, 5, 0)\})$),
- $t(S, n_2) = 8$ (reached in $d((8, 0, 0), \{(0, 5, 1)\})$),
- $t(S, n_2) = 6$ (reached in $d((0, 10, 0), \{(1, 0, 8), (2, 5, 3), (5, 5, 1), (7, 0, 4), (10, 0, 2)\})$).

Thus $t(S) = 8$ and this is an example where $c(S) < t(S)$. □

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